STRONG ERGODICITY AND QUOTIENTS OF EQUIVALENCE RELATIONS

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1. STRONG ERGODICITY

Throughout this note $(X,\mathcal{S},\mu)$ will be a standard, nonatomic probability space. Let G be a countable group, and let $(g,x) \to T_g x$ be a nonsingular, ergodic action of G on $(X,\mathcal{S},\mu)$. A sequence $(B_n) \subset \mathcal{S}$ is <u>asymptotically invariant</u> (a.i.) under the action T of G if $\lim_n \mu(B_n \Delta T_g B_n) = 0$ for every $g \in G$, and (B_n) is <u>trivial</u> if $\lim_n \mu(B_n).(1-\mu(B_n)) = 0$. The action of G on $(X,\mathcal{S},\mu)$ is <u>strongly ergodic</u> if every a.i. sequence is trivial.

The <u>full group</u> $[T]$ of the action T of G on $(X,\mathcal{S},\mu)$ is the group of all nonsingular automorphisms V of $(X,\mathcal{S},\mu)$ such that $Vx \in T_G x = \{T_g x : g \in G\}$ for μ-a.e. $x \in X$. The following assertion is elementary and implies that strong ergodicity is a property of the full group $[T]$ or, equivalently, a property of the equivalence relation of T (cf. section 2).

1.1 PROPOSITION [6] *Let* (B_n) *be an a.i. sequence for* T. *Then* $\lim_n \mu(B_n \Delta V B_n) = 0$ *for every* $V \in [T]$.

1.2 EXAMPLE [6] Let V be a measure preserving, ergodic automorphism of a probability space $(X,\mathcal{S},\mu)$. Rokhlin's lemma implies that there exists, for every $n \geq 1$, a set $C_n \in \mathcal{S}$ such that $\mu(C_n) = \frac{1}{2n}$ and $C_n \cap V^k C_n = \phi$ for $1 \leq k \leq 2n-2$. Put $B_n = \bigcup_{k=0}^{n-1} V^k C_n$ and observe that