LIE GROUP IMBEDDINGS OF THE FOURIER TRANSFORM AND A NEW FAMILY OF UNCERTAINTY PRINCIPLES

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1. INTRODUCTION

The one-dimensional Fourier-Plancherel operator $F: L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined formally by

$$(1.1) \qquad (Ff)(y) = \hat{f}(y) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iyx} f(x)\,dx,$$

is a unitary operator; that is

$$(1.2) \qquad \langle f,g \rangle = \langle \hat{f},\hat{g} \rangle \text{ and } \|f\| = \|\hat{f}\|$$

where

$$(1.3) \qquad \langle f,g \rangle = (2\pi)^{-1/2} \int_{\mathbb{R}} \bar{f}(x) g(x)\,dx \text{ and } \|f\| = \langle f,f \rangle^{1/2};$$

also $F^4 = I$, the identity operator, so the integer powers of F form a cyclic group of order 4 [5]. It is natural to contemplate imbedding this finite discrete group of unitary operators in a continuous one. Condon derived a one-parameter group of integral operators $\{F_\theta\}$ ($\theta \in \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$) with the appropriate properties in 1937 [2] and Bargmann derived a corresponding one-parameter group for the d-dimensional Fourier operator in 1961 [1]. I have shown [11] the construction of infinitely many distinct imbeddings of the d-dimensional