DISCREPANCY RESULTS FOR NORMAL NUMBERS

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1. INTRODUCTION

We say that a number x is *normal to base* r ($r \in \mathbb{Z}^+$) if the sequence $(r^n x)_{n=1}^{\infty}$ is uniformly distributed modulo unity. Weyl [16] established that for each base r almost all (Lebesgue) numbers are normal. The nature of the set of numbers normal to all bases in some collection A and non-normal to every base of a collection B was first investigated by W. Schmidt [14,15]. As observed by Schmidt, such numbers can arise if and only if there is no relation of the form $r^n = s^m$ for $r \in A, s \in B$ and $m, n \in \mathbb{Z}^+$. In this event we say A, B are *multiplicatively independent*. We assume this restriction without further comment. Later refinements and some simpler proofs have been given in [1,2,12]. In [2] it was shown that for an appropriately chosen Riesz product measure μ on $[0,1]$, the numbers on $[0,1]$ which are normal to each base of A and normal to no base of B constitute a set of full μ measure. It is seen further in [3] that this set has Hausdorff dimension unity.

Our present purpose is to derive some corresponding discrepancy results pertaining to the rate of convergence to uniformity of the sequences involved. For simplicity we restrict our attention to the case considered in [1] in which B is a singleton set $\{s\}$ with $s > 3$, but the arguments go through with appropriate modifications in the rather more general situation of [2]. The proof employed is based on estimates of Fourier-Stieltjes coefficients available from earlier studies together with some elementary probability theory.

To describe the rate of convergence towards uniformity of a sequence $(x_n)_1^{\infty}$ of numbers on $[0,1]$ we recall the notion of discrepancy. For $E \subset [0,1]$, we define the *counting function*

$$A(E;n) \equiv \text{card}\{k | \quad 0 < k \leq n, \quad x_k \in E\} \quad .$$