SINGULAR INTEGRALS ON BMO

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Let f be a locally integrable function on $\mathbb{R}^n$. We say f has bounded mean oscillation, $f \in BMO$, if

$$(1) \qquad \sup_B \inf_{c\in\mathbb{R}} \frac{1}{|B|} \int_B |f(y)-c|\,dy < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. Identifying functions which differ by an additive constant a.e. makes BMO a Banach space with norm $\| \; \|_{BMO}$ equal to the left hand side of (1). Note that L^∞ is a proper subset of BMO, since $\log|x| \in BMO$.

Let K be a locally integrable function on $\mathbb{R}^n\backslash\{0\}$ such that $Tf(x) = \lim_{\epsilon\downarrow 0} \int_{\{|y|>\epsilon\}} K(y)f(x-y)dy$ is a bounded operator on L^2. We say K satisfies condition H_r, $1 \le r < \infty$, if there is a non–decreasing function s on $(0,1)$ such that $\sum_{j=1}^\infty s(2^{-j}) < +\infty$ and

$$\left[\int_{\{x:R<|x|<2R\}} |K(x-y)-K(x)|^r\,dx\right]^{1/r} \le s\left(\frac{|y|}{R}\right) R^{-n/r'}, \text{ for } |y|<R/2.$$

Define H_∞ by the obvious modification.

If $f \in L^\infty$ is supported on a set of finite measure and $K \in H_1$, then Tf exists a.e. (i.e., the limit exists and is finite), $Tf \in BMO$, and $\|Tf\|_{BMO} \le C\|f\|_{BMO}$ [2]. On the other hand, if f is merely bounded, then without a suitable modification Tf may fail to exist on a set of positive measure. For example, if $f(x) = \chi_E(x)$ is the characteristic function of $E = \{x \in \mathbb{R}^n: x_i > 0, i=1,\dots,n\}$, then the Riesz transforms of f, defined by the kernels $K_j(x) = \dfrac{x_j}{|x|^{n+1}}$, $j=1,\dots,n$, are infinite a.e.