A QUALITATIVE UNCERTAINTY PRINCIPLE FOR LOCALLY COMPACT ABELIAN GROUPS

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1. INTRODUCTION

It has long been known that if a function $f \in L^2(\mathbb{R}^n)$ and the supports of f and its Fourier transform $\hat{f}$ are contained in bounded rectangles, then $f = 0$ almost everywhere. In 1974, Benedicks [2] strengthened this result by showing that the supports of f and $\hat{f}$ having finite measure is sufficient to imply that $f = 0$ almost everywhere. Amrein and Berthier [1] reached the same conclusion in 1977 using Hilbert space methods. This result may be thought of as a qualitative uncertainty principle since it limits the "concentration" of the Fourier transform pair $(f,\hat{f})$. Little is known, however, of analogous behaviour for functions on locally compact abelian (LCA) groups.

Let G be an LCA group with dual group Γ. Equip G with a Haar measure m_G. If $\gamma \in \Gamma$ and $f \in L^1(G)$, the Fourier transform $\hat{f}$ of f is given by

$$\hat{f}(\gamma) = \int_G f(x)\,\overline{\gamma(x)}\,dm_G(x).$$

We choose a Haar measure m_Γ on Γ for which the Plancherel identity is valid.

If G is compact, then Γ is discrete and we insist $m_G(G) = 1$. With this convention, $m_\Gamma(\{0\}) = 1$.