RIGIDITY, VON NEUMANN ALGEBRAS, AND SEMISIMPLE GROUPS

by Michael Cowling

The material described below is largely contained in three papers, by U. Haagerup [H], M. Cowling and U. Haagerup [CH], and by M. Cowling and R.J. Zimmer [CZ].

1 WHAT IS RIGIDITY? G. Mostow, with a little help from his friends, proved the following result, known as Mostow's rigidity theorem.

THEOREM. *Let Γ_1 and Γ_2 be lattices in the centreless semisimple Lie groups G_1 and G_2 respectively (i.e. Γ_i is a discrete subgroup of G_i and the homogeneous space G_i/Γ_i has finite G_i-invariant measure, for $i = 1,2$). Any isomorphism $\Gamma_1 \to \Gamma_2$ extends to an isomorphism $G_1 \to G_2$.*

This theorem tells us lattices in nonisomorphic semisimple Lie groups cannot be isomorphic. Subsequently, G. Margulis extended this theorem to show that any homomorphism $\Gamma_1 \to \Gamma_2$ extends to a homomorphism $G_1 \to G_2$, provided that G_1 and G_2 are of real rank at least 2, and Γ_1 and Γ_2 are irreducible lattices. This extension, known as Margulis' super-rigidity theorem, gives more information about how different Γ_1 and Γ_2 must be for different G_1 and G_2. The geometric interest of such theorems lies in the fact that the groups Γ arise naturally as the fundamental groups of certain locally symmetric spaces (of the form $K\backslash G/\Gamma$); one deduces that if the spaces are locally different, in their differential geometric structure, then they are globally different, topologically.

A rigidity theorem, then, is one which tells us that objects — in particular lattices — are different.

There are several ways to look at the question of how groups differ, including the von Neumann algebraic and the ergodic theoretic viewpoints.

In studying groups, one frequently encounters the von Neumann algebra $VN(G)$ of the group G. This is the algebra of all bounded operators on $L^2(G)$ (relative to a left-invariant Haar measure) which commute with right translations. If G is abelian, then $VN(G)$ is isomorphic to $L^\infty(\hat{G})$, where $\hat{G}$ is the Pontryagin dual group of G. Since there are nonisomorphic groups with the same von Neumann algebras (for instance, C_4 and $C_2 \times C_2$, where C_n is the cyclic group of order n), but