6. SCALAR OPERATORS

The most important applications of integration with respect to Banach space valued measures undoubtedly arise in the theory of spectral operators. To describe its central notion, let E be a complex Banach space, $\mathrm{BL}(E)$ the algebra of all bounded linear operators on E and I the identity operator. A spectral measure is an additive and multiplicative map $P : \mathcal{Q} \to \mathrm{BL}(E)$, whose domain, $\mathcal{Q}$, is an algebra of sets in a space Ω, such that $P(\Omega) = I$. An operator $T \in \mathrm{BL}(E)$ is said to be of scalar type if there exists a σ-additive (in the strong operator topology) spectral measure, P, whose domain is a σ-algebra and a P-integrable function f such that

$$T = \int_\Omega f \mathrm{d}P. \qquad (*)$$

This notion, due to N. Dunford, extends to arbitrary Banach space the idea of an operator with diagonalizable matrix on a finite-dimensional space. It proved to be very fruitful as shows the exposition in Part III of the monograph [14]. Many powerful techniques in which scalar operators play a role are based on the requirements that $\mathcal{Q}$ be a σ-algebra and that P be σ-additive. But precisely these requirements are responsible for excluding many operators of prime interest from the class of scalar-type operators.

In this chapter, we present a suggestion for extending this class, [35]. It is based on the fact that the integral (*) exists if and only if there exist $\mathcal{Q}$-simple functions f_j, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} \| \int_\Omega f_j \mathrm{d}P \| < \infty$$

and the equality

$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

holds for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$