5. VECTOR VALUED FUNCTIONS AND PRODUCTS

The title practically gives away the content of this chapter. We present first a Bochner-type integration theory, that is, one based on absolute summability, for Banach space valued functions. Then we consider direct products of integrating gauges along with the corresponding Fubini- and Tonelli-type theorems. These two themes are related in the formulation of the mentioned theorems; the notion of a measurable function is avoided by stating them in terms of Bochner integrability.

A. Let ρ be a gauge on a nontrivial family, $\mathcal{K}$, of scalar valued functions on a space Ω. (See Section 2A.)

Let E be a Banach space. To avoid some obvious trivialities, we assume that E contains a non-zero vector. The convention of writing interchangeably $ca = ac$, for every $c \in E$ and a scalar a, will be used throughout the chapter.

A function $f : \Omega \to E$ will be called Bochner integrable with respect to ρ, or, briefly, ρ-integrable, if there exist vectors $c_j \in E$ and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, such that

(A.1) $$\sum_{j=1}^{\infty} |c_j| \rho(f_j) < \infty$$

and

(A.2) $$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for every $\omega \in \Omega$ for which

(A.3) $$\sum_{j=1}^{\infty} |c_j| \, |f_j(\omega)| < \infty .$$

The family of all E-valued functions on Ω, Bochner integrable with respect to ρ, is denoted by $\mathcal{L}(\rho,\mathcal{K},E)$. If the space E happens to be one-dimensional, that is, just the space of scalars, then, consistently with the notation introduced in Chapter 2, we write $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{L}(\rho,\mathcal{K},E)$.