4. SET FUNCTIONS

Given an additive set function, μ, on a semiring of sets, $\mathcal{Q}$, the problem arises naturally of finding a gauge which integrates for μ. (See Section 3A.) If there exists a finite non-negative σ-additive set function, ι, on $\mathcal{Q}$ such that $|\mu(X)| \leq \iota(X)$, for every $X \in \mathcal{Q}$, then μ is said to have finite variation. In that case, ι is a gauge integrating for μ. This situation is classical.

The point of this chapter is that, even when μ does not have finite variation, there may exist gauges integrating for μ. For, there may exist a continuous, convex and increasing function, Φ, on $[0,\infty)$ such that $\Phi(0) = 0$ and a σ-additive set function $\iota : \mathcal{Q} \to [0,\infty)$ such that $\Phi(|\mu(X)|) \leq \iota(X)$, for every $X \in \mathcal{Q}$. Then $|\mu(X)| \leq \rho(X)$, where $\rho(X) = \varphi(\iota(X))$, for every $X \in \mathcal{Q}$, and φ is the inverse function to Φ. By Proposition 2.26, the gauge ρ is integrating.

So, we are led to the consideration of higher variations introduced by N. Wiener and L.C. Young. (See Example 4.1 in Section A below.)

A. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. Recall that, by $\Sigma = \Sigma(\mathcal{Q})$ is denoted the set of all families of pair-wise disjoint sets belonging to $\mathcal{Q}$. (See Section 1D.) An element, $\mathcal{P}$, of Σ such that its union is equal to Ω and, for every $X \in \mathcal{Q}$, the sub-family $\{Y \in \mathcal{P} : Y \cap X \neq \emptyset\}$ of $\mathcal{P}$ is finite, is called a partition. The set of all partitions is denoted by $\Pi = \Pi(\mathcal{Q})$.

Let E be a Banach space and $\mu : \mathcal{Q} \to E$ an additive set function.

Given a Young function Φ (see Section 1G), a set X from $\mathcal{Q}$ and a partition $\mathcal{P}$, let

(A.1) $$v_{\Phi}(\mu,\mathcal{P};X) = \sum_{Y \in \mathcal{P}} \Phi(|\mu(X \cap Y)|) .$$

Then, for the given Φ, X and a set of partitions $\Delta \subset \Pi$, let

(A.2) $$v_{\Phi}(\mu,\Delta;X) = \sup\{ v_{\Phi}(\mu,\mathcal{P};X) : \mathcal{P} \in \Delta\} .$$