3. INTEGRALS

Besides an integrating gauge, ρ, on a family of functions, $\mathcal{K}$, we consider a functional, μ, on $\mathcal{K}$ which can be extended to a continuous linear functional, μ_ρ, defined on the whole of $\mathcal{L} = \mathcal{L}(\rho,\mathcal{K})$. The continuity is understood with respect to the seminorm, q_ρ, induced by ρ on $\mathcal{L}$, defined in the previous chapter. More generally, we consider a map, μ, from $\mathcal{K}$ into an arbitrary Banach space, E, and a continuous linear map, μ_ρ, from $\mathcal{L}$ into E, generated by μ. Given a function $f \in \mathcal{L}$, the number, or vector, $\mu_\rho(f)$ is looked upon as the integral of f with respect to μ.

The classical case of integration with respect to a (positive) measure, ι, is obtained by taking for $\mathcal{K}$ a sufficiently rich family of (characteristic functions of) sets of finite measure and putting both ρ and μ equal to (the restriction to $\mathcal{K}$ of) ι. If μ is an additive set function having finite and σ-additive variation, then integration with respect to μ can be introduced by choosing ρ equal to the variation of μ. Of course, this choice is not available in general, and so, given an additive set function, μ, the problem of integration with respect to μ is reduced to that of finding a suitable ρ. This problem will be treated more systematically in Chapter 4.

Here we show how the integration with respect to Banach space valued measures, due to R.G. Bartle, N. Dunford and J.T. Schwartz, [2], fits into the presented scheme. Also in this chapter, the definitions of the Orlicz, the Sobolev and the Hardy spaces are shown to be special cases of the construction of the space $\mathcal{L}(\rho,\mathcal{K})$ for suitable choices of $\mathcal{K}$ and ρ.

A. Let $\mathcal{K}$ be a nontrivial family of functions on a space Ω. Let E be a Banach space. Let $\mu : \mathcal{K} \to E$ be a linear map. Recall that the domain of a linear map, or a linear functional, is not necessarily a vector space. (See Section 1E.)

We shall say that a gauge, ρ, on $\mathcal{K}$ integrates for the map μ if it is integrating (see Section 2D) and $|\mu(f)| \leq c q_\rho(f)$, for some number $c \geq 0$ and every function $f \in \mathrm{sim}(\mathcal{K})$.