THE NORMS OF POWERS OF FUNCTIONS IN THE VOLTERRA ALGEBRA, II

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In this note we provide an example of a weight sequence $\{\omega_n\}_{n\geq 1}$ which satisfies (i) $\omega_n \geq 0$, (ii) $\omega_{n+m} \leq \omega_n \omega_m$, (iii) $\omega_n^{1/n} \to 0$, and (iv) $\omega_n^{1/n}$ is monotone decreasing, but for which there is no positive $\mu \in (L^1[0,1],*)$ with $\omega_n = \|\mu^n\|$ for every n. This answers the problem of [1], whereas, as detailed there, the example of [2] is for a different, albeit related, problem.

LEMMA. *If* $\mu \in (L^1[0,1],*)$ *is positive and non-nilpotent, then* $\dfrac{\|\mu^{2n}\|}{\|\mu^n\|^2} \to 0$ *as* $n \to \infty$.

Proof. It is shown in [1] that $\|\mu^n\|^{\frac{1}{n}}$ is monotone decreasing. Hence,

$$\frac{\|\mu^{n+1}\|}{\|\mu^n\|} = \frac{(\|\mu^{n+1}\|^{\frac{1}{n+1}})^{n+1}}{(\|\mu^n\|^{1/n})^n}$$

$$= \left[\frac{\|\mu^{n+1}\|^{\frac{1}{n+1}}}{\|\mu^n\|^{1/n}}\right]^n \cdot \|\mu^{n+1}\|^{\frac{1}{n+1}}$$

$$\leq \|\mu^{n+1}\|^{\frac{1}{n+1}} \to 0 \text{ as } n \to \infty,$$

that is, the sequence $(\|\mu^n\|)_{n=1}^{\infty}$ is regulated.

Now let $J = \{f \in L^1[0,1]: \lim_{n\to\infty} \frac{\|f*\mu^n\|}{\|\mu^n\|} = 0\}$. Then J is a closed ideal in $(L^1[0,1],*)$, [2]. Since μ is not nilpotent, $\inf(\mathrm{supp}(\mu)) = 0$, and since $\mu \in J$ it follows that $J = L^1[0,1]$. Therefore, $\lim_{n\to\infty} \frac{\|f*\mu^n\|}{\|\mu^n\|} = 0$ for every $f \in L^1[0,1]$. If p is a probability measure with support contained in $(0,\frac{1}{2})$, then $\|p*\mu^n\| \geq \|\delta_{1/2}*\mu^n\|$ and so

$$\lim_{n\to\infty} \frac{\|\delta_{1/2}*\mu^n\|}{\|\mu^n\|} = 0.$$