ULTRAPRIME GROUP ALGEBRAS

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Let $\mathcal{A}$ be a Banach algebra and for each pair, (a,b), of elements of $\mathcal{A}$ define a map $M_{a,b} : \mathcal{A} \to \mathcal{A}$ by $M_{a,b}(x) = axb$. Then $\mathcal{A}$ is said to be *prime* if $M_{a,b} \neq 0$ whenever a and b are non-zero, and to be *ultraprime* if there is a constant, $K > 0$, such that $\|M_{a,b}\| \geq K\|a\|\|b\|$ for every (a,b), see [5]. The centre of $\mathcal{A}$ will be denoted $Z(\mathcal{A})$.

Let G be a discrete group and let $\ell^1(G)$ denote its group algebra. It is easily checked that a function, f, belongs to $Z(\ell^1(G))$ if and only if f is constant on the conjugacy classes of G, that is, if and only if $f(x) = f(yxy^{-1})$ for every $x,y \in G$. Hence δ_e, the point mass at the identity element, belongs to $Z(\ell^1(G))$, and $Z(\ell^1(G))$ has dimension greater than one if and only if G has a finite conjugacy class other than {e}. These observations will be useful in the proof of the following.

PROPOSITION. (i) *If* $Z(\ell^1(G))$ *has dimension greater than one, then* $\ell^1(G)$ *is not prime.*

(ii) *If the dimension of* $Z(\ell^1(G))$ *equals one then* $\ell^1(G)$ *is ultraprime.*

Proof. (i) $Z(\ell^1(G))$ is semisimple because it has a faithful $*$-representation on $\ell^2(G)$. Hence, if $Z(\ell^1(G))$ has dimension greater than one, then there are two distinct points, p and q, in the space X of multiplicative linear functionals on $Z(\ell^1(G))$. Since X is Hausdorff there are open sets $U,V \subset X$ such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$. By Theorem 1.8 in [4], $Z(\ell^1(G))$ is a regular Banach algebra and so there exist $a,b \in Z(\ell^1(G))$ such that $\hat{a}(p) = 1$ and $\hat{a}$ is zero outside U, $\hat{b}(q) = 1$ and $\hat{b}$ is zero outside V. Hence a and b are non-zero and $ab = 0$. Since $a,b \in Z(\ell^1(G))$, $M_{a,b} = 0$ and so $\ell^1(G)$ is not prime.

(ii) The second part will be proved in a sequence of lemmas.