SUBALGEBRAS OF AMENABLE ALGEBRAS

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It is well known that a closed ideal J of an amenable Banach algebra A is itself amenable if and only if it has a bounded approximate identity, and that if J and A/J are both amenable, then so is A. In this note we give some remarks concerning the question of whether or not anything can be said about closed subalgebras of amenable Banach algebras.

We begin by recalling some results from harmonic analysis which show, in various guises, that the "standard" amenable algebras, $L^1(\Gamma)$, for locally compact abelian groups Γ, always contain non-amenable closed subalgebras, provided that Γ is infinite. For convenience of notation, our discussion will be in terms of the dual group G, and the Fourier algebra $A(G) = L^1(\Gamma)^\wedge$.

Looking firstly at closed ideals, the remarks above show that the following gives a chacterization of amenable ideals in $A(G)$.

THEOREM 1 ([17], Theorem 13). *Let G be a locally compact abelian group, $I \subseteq A(G)$ a closed ideal. Then I has a bounded approximate identity if and only if $I = I(E) = \{ f \in A(G) : f(E) = (0) \}$ for some closed set $E \subseteq G$ which lies in the coset ring of G considered as a discrete group.*

In particular, if I has a bounded approximate identity then $E = hull(I)$ is a set of synthesis (though not conversely). Thus for G infinite and non-discrete, Malliavin's theorem shows the existence of E with $I(E)$ not amenable. Again, for G infinite and compact, spectral synthesis fails spectacularly in the sense that there exists $f \in A(G)$ such that the closed ideals generated by the positive powers of f are all distinct ([19]); of course these ideals cannot have bounded approximate identities. For G discrete, so every subset of G is of synthesis, take

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