REMARKS ON SEMI-SIMPLE REFLEXIVE ALGEBRAS

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1. INTRODUCTION AND PRELIMINARIES

Below, we discuss the problem: For which subspace lattices $\mathcal{L}$ on a complex Banach space X is $\operatorname{Alg}\mathcal{L}$ semi-simple? Our first proposition gives a lattice-theoretic necessary condition on $\mathcal{L}$ in order that $\operatorname{Alg}\mathcal{L}$ be semi-simple, and an improvement of a result of Lambrou [5] follows from it. The converse of this proposition, though not true in general, is shown to be valid for two large classes of subspace lattices, each of which contains every completely distributive subspace lattice amongst its members. Each of these two classes also contains every pentagon. The Alg of every pentagon is shown to be semi-simple. For certain subspace lattices which belong to both these aforementioned classes we obtain a result (Theorem 3) which has another result of Lambrou [5] as a corollary. Examples are given to show that, in general, the semi-simplicity of $\operatorname{Alg}\mathcal{L}$ cannot be a purely lattice-theoretic property of $\mathcal{L}$; more precisely, we show that it is possible to have $\mathcal{L}_1$ and $\mathcal{L}_2$ lattice-isomorphic with $\operatorname{Alg}\mathcal{L}_1$ semi-simple and $\operatorname{Alg}\mathcal{L}_2$ not. We also show that any double triangle on X with elements $K, L \neq X$ satisfying $K + L = X$ has a semi-simple Alg.

We use the notation, terminology and definitions of the preceding article [10]. Some of these are repeated here for the convenience of the reader. In particular, throughout X denotes a complex non-zero Banach space. Also, by a *subspace lattice on X* we mean a collection $\mathcal{L}$ of subspaces of X satisfying (i) $(0), X \in \mathcal{L}$, and (ii) for every family $\{M_\gamma\}$ of elements of $\mathcal{L}$, both $\cap M_\gamma$ and $\vee M_\gamma$ belong to $\mathcal{L}$. For every subspace lattice $\mathcal{L}$ on X we define $\operatorname{Alg}\mathcal{L}$ by

$$\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(X) : TM \subseteq M\,, \quad \text{for every } M \in \mathcal{L}\}\,.$$

For every $\mathcal{L}$, $\operatorname{Alg}\mathcal{L}$ is a unital Banach algebra and the class of operator algebras of the form $\operatorname{Alg}\mathcal{L}$ for some subspace lattice $\mathcal{L}$ (on X) is the class of *reflexive* operator algebras