# USING BANACH ALGEBRAS TO DO ANALYSIS WITH THE UMBRAL CALCULUS

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## 1. INTRODUCTION

The modern version of the umbral calculus, which was developed by G.C. Rota with a number of collaborators, particularly Steven Roman, has roots which go far back into the nineteenth century. According to them, [13, Section 1, pp.95–98], the umbral calculus was originally an attempt to exploit the fact that one could often substitute appropriate sequences of numbers or polynomials for the sequence $\{x^n\}$ of powers in certain identities. For instance, many polynomial sequences $\{p_n(x)\}$ satisfy the binomial identity

$$p_n(x+y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} p_k(y)\, p_{n-k}(x) . \tag{1.1}$$

By the early twentieth century, it was clear that the umbral calculus was closely tied to the Heaviside operational calculus. Recall that when we let $t$ be the differentiation operator $d/dx$, then for the power series $f(t) = \Sigma_0^\infty a_n t^n$ the operational calculus formula is

$$f(t)\, h(x) = \Sigma a_n\, h^{(n)}(x) , \tag{1.2}$$

so that, in particular, $e^{at} h(x) = h(x+a)$ .

Throughout the nineteenth century, both the umbral calculus and the operational calculus were powerful heuristic devices for discovering useful formulae, but these formulae needed to then be rigorously demonstrated by other means. While there have been many successful rigorous versions of the operational calculus, for instance the elegant Mikusiński calculus, [9], the umbral calculus resisted being made rigorous until relatively recently. In 1970, in a path–breaking paper [10], Mullin and Rota developed a rigorous