ON NORMS ON ALGEBRAS

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1. INTRODUCTION

In order to be quite explicit, let me collect in this section some fundamental definitions.

Throughout, an *algebra* is a linear, associative algebra over a field $\mathbb{F}$, which is always either $\mathbb{R}$ or $\mathbb{C}$. An algebra is *unital* if it has an identity; the identity is usually denoted by e. If A is an algebra without identity, then $A^{\#}$ is the algebra formed by adjoining an identity to A; if A is unital, then $A^{\#} = A$.

1.1. DEFINITION. Let A be an algebra. An *algebra seminorm* on A is a map $\|.\|:A \to \mathbb{R}$ such that:

(i) $\|a\| \geq 0 \quad (a \in A)$;

(ii) $\|a + b\| \leq \|a\| + \|b\| \quad (a,b \in A)$;

(iii) $\|\alpha a\| = |\alpha| \, \|a\| \quad (\alpha \in \mathbb{F}, a \in A)$;

(iv) $\|ab\| \leq \|a\| \, \|b\| \quad (a,b \in A)$.

The map $\|.\|$ is an *algebra norm* if, further:

(v) $\|a\| > 0 \quad (a \in A \setminus \{0\})$.

An algebra A with a norm $\|.\|$ is a *normed algebra*; the normed algebra is a *Banach algebra* if the norm is complete. An algebra A is *normable* if there is an algebra norm on A.

The *spectrum* $\sigma(a)$ of an element a of an algebra A is

$$\sigma(a) = \{z \in \mathbb{C} : ze - a \text{ is not invertible in } A^{\#}\},$$

and the *spectral radius* of a is