COMPLEMENTATION PROBLEMS CONCERNING THE RADICAL OF A COMMUTATIVE AMENABLE BANACH ALGEBRA

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In 1986 Bachelis and Saeki, [1], showed that if $\mathfrak{A}$ is a commutative Banach alegbra, with identity and non-zero radical R, which in addition satisfies the following condition

$$A : \mathrm{sp}\{x \in \mathfrak{A}^{-1} : \sup_{n\in\mathbb{Z}} \|x^n\| < \infty\}^- = \mathfrak{A},$$

then there does not exist a closed subalgebra $\mathfrak{B}$ complementary to the radical R (or complementary to any closed ideal I of $\mathfrak{A}$ contained in R).

In [2] R. J. Loy and the present author extended these results in the following way to commutative Banach algebras satisfying either of the following weaker generating conditions.

$$B : \mathrm{sp}\{x \in \mathfrak{A}^{-1} : \|x^n\|\|x^{-n}\| = o(n)\}^- = \mathfrak{A}$$

$$C : \mathrm{sp}\{x \in \mathfrak{A} : \|e^{nx}\|\|e^{-nx}\| = o(n)\}^- = \mathfrak{A}$$

THEOREM 1. *Let $\mathfrak{A}$ be a commutative Banach algebra with identity which satisfies either of the condition B or C. If φ and ψ are continuous homomorphisms of $\mathfrak{A}$ into the commutative Banach algebra $\mathfrak{B}$ such that*

$$(\varphi-\psi)(\mathfrak{A}) \subset \mathit{rad}\,\mathfrak{B},$$

then $\varphi = \psi$.

It follows immediately that if $\mathfrak{A}$ is commutative satisfying B or C, and rad $\mathfrak{A} \equiv R \neq 0$, then $\mathfrak{A}$ cannot have the strong Wedderburn property, that is, there cannot exist a closed subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ with $\mathfrak{C} \simeq \mathfrak{A}/R$ and $\mathfrak{A} = \mathfrak{C} \oplus R$. A similar result holds if I is any closed ideal of $\mathfrak{A}$ contained in R. On the other hand, if $\mathfrak{B}$ is a commutative Banach algebra which satisfies $\mathfrak{B} = \mathfrak{C} \oplus I$, where $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{B}$ continuously isomorphic to $\mathfrak{A}$, and I is a closed ideal of $\mathfrak{B}$