POWER-BOUNDED ELEMENTS IN A BANACH ALGEBRA AND A THEOREM OF GELFAND

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1. INTRODUCTION

Most of the unattributed results in §§1–4 are joint work with T.J. Ransford; a fully detailed account, including results not given here, is in [2].

Let A be a (complex) unital Banach algebra and let $x \in A$. We say that:

(i) x is *power-bounded* (pb) if and only if there is $K > 0$ such that $\|x^n\| \leq K$ for all $n \geq 0$;

(ii) x is *doubly power-bounded* (dpb) if and only if x is invertible and there is $K > 0$ such that $\|x^n\| \leq K$ for all $n \in \mathbf{Z}$.

We remark that we may always renorm with an equivalent unital algebra norm so that $K = 1$.

From the spectral radius formula: *if x is pb then $Sp\,x \subseteq \Delta \equiv \{z \in \mathbf{C} : |z| \leq 1\}$; if x is dpb then $Sp\,x \subseteq \mathbf{T} \equiv \{z \in \mathbf{C} : |z| = 1\}$.*

We shall give a very simple proof of the following result of Katznelson and Tzafriri [10].

THEOREM 1. (Katznelson & Tzafriri) *Let A be a complex unital Banach algebra and let $x \in A$ be pb. Then $\|x^n(1-x)\| \to 0$ as $n \to \infty$ if (and only if) $(Sp\,x) \cap \mathbf{T} \subseteq \{1\}$.*

Remarks. (i) The 'only if' is trivial.

(ii) If $(Sp\,x) \cap \mathbf{T} = \emptyset$, then $x^n \to 0$, since $r_A(x) < 1$, so the only case of interest is that in which $(Spx) \cap \mathbf{T} = \{1\}$.