MONOTONE OPERATORS AND OBSTACLE PROBLEMS

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1. INTRODUCTION

Let Ω be a bounded open set in R^n and let ψ be a continuous function on $\partial\Omega$ (the boundary of Ω). Consider first of all the following problem.

Does there exist a continuous function u on the closure $\bar{\Omega}$ of Ω, agreeing with ψ on $\partial\Omega$, C'' on Ω and satisfying the non-linear partial differential equation

$$(1) \qquad \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u(x), Du(x)) - b(x, u(x), Du(x)) = 0$$

for $x \in \Omega$. The a_i and b are given functions.

If the a_i and b are sufficiently smooth and satisfy an ellipticity condition as well as certain growth conditions and if $\partial\Omega$ is sufficiently smooth, then such a function u is known to exist. One way of proving this is to use fixed point theory.

When such a function u exists and ϕ is a suitably smooth function, vanishing on $\partial\Omega$, then repeated integration and integration by parts gives

$$(2) \qquad \sum_{i=1}^{n} \int_\Omega a_i(x, u, Du) \frac{\partial \phi}{\partial x_i} dx + \int_\Omega b(x, u, Du) \phi dx = 0.$$

Equation (2) still makes sense when the a_i, b and u are less smooth. u is called a weak solution when (2) holds for all ϕ.

When showing that a weak solution exists (under appropriate conditions on a_i, b and $\partial\Omega$), fixed point theory does not seem to work. But a valuable tool is provided by the theory of monotone and pseudo-monotone operators. This theory is based on the following two results in R^n.

(A) The Brouwer fixed point theorem.

(B) If K is a compact convex, non-empty subset of R^n and $x \in R^n \sim K$, then there is a $y \in K$, such that

$$(u - y) \cdot (y - x) \geq 0$$

for all $u \in K$.