A Note on Martingales with respect to Complex Measures

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Introduction

Let γ be a rectifiable Jordan curve passing through ∞, and let $z(x)$ denote its arclength parameterization. Assume that γ is a chord-arc curve: this means that there is a constant such that

$$1 \leq \frac{|a-b|}{|\int_a^b z'(x)dx|} = \frac{|a-b|}{|z(a)-z(b)|} \leq C_0 < \infty, \quad \forall a,\ b.$$

Let $\mathcal{D}_k$ denote the ring of sets generated by the collection of dyadic intervals of length 2^{-k}, $k \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, and define the "conditional expectation" operator E_k by

$$E_k f(t) = \int_I f(x)z'(x)dx \Big/ \int_I z'(x)dx, \quad t \in I,$$

where I is a dyadic interval of length 2^{-k}. The operator E_k has a natural extension to $\mathcal{D}_k$. It may be thought of, in a natural way, as a conditional expectation with respect to the finitely-additive complex measure $z'(x)dx$. In a recent paper, Coifman, Jones and Semmes [CJS] pointed out that this conditional expectation operator has many of the same properties as the conditional expectation with respect to a positive measure. They outlined a proof of the corresponding Littlewood-Paley theorem which made use of a Carleson measure argument, and used the Littlewood-Paley theorem to give a new proof of the L^2-boundedness of Cauchy integrals along chord-arc curves.

In this note we establish a general theory of martingales with respect to complex measures. In our case, the complex measures are defined and σ-additive on a σ-algebra

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