32 Appendix The Eigenvalue Problem

In order to discuss the stabilities of a minimal surface, we need some general knowledge of the (Dirichlet) eigenvalues of a self-adjoint second order elliptic operator.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and L be a self-adjoint second order elliptic operator

$$Lu = D_i(a^{ij}D_ju + b^iu) - b^iD_iu + cu,$$

where (a^{ij}) is symmetric. We suppose that L satisfies

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbf{R}^n, \tag{32.186}$$

$$\sum |a^{ij}(x)|^2 \leq \Lambda^2, \quad 2\lambda^{-2}\sum_{i=1}^{n}\left(|b^i(x)|^2 + \lambda^{-1}|c(x)|\right) \leq \nu^2, \quad \forall x \in \Omega, \tag{32.187}$$

for some constants λ, Λ, $\nu > 0$.

Define $(u,v) = \int_\Omega uv\,dx$, and a quadratic form on $H = H(\Omega) = W_0^{1,2}(\Omega)$ by

$$\mathcal{L}(u,v) = \int_\Omega (a^{ij}D_iuD_jv + b^iuD_iv + b^ivD_iu - cuv)dx = -(Lu,v).$$

The ratio

$$J(u) = \frac{\mathcal{L}(u,u)}{(u,u)}, \quad u \not\equiv 0, \quad u \in H,$$

is called the Rayleigh quotient of L.

By (32.186) and (32.187) we see that J is bounded from below. In fact, writing $b = (b^1, \cdots, b^n)$ and $|b|^2 = \sum_i |b^i|^2$, we have

$$\begin{aligned}
\mathcal{L}(u,u) &= \int_\Omega (a^{ij}D_iuD_ju + 2b^iuD_iu - cu^2)dx \\
&\geq \int_\Omega \left[\lambda|Du|^2 - \left(\frac{1}{2}\lambda|Du|^2 + 2\lambda^{-1}|b|^2u^2 + cu^2\right)\right]dx \\
&\quad \text{(by (32.186) and Schwarz's inequality)} \\
&\geq \int_\Omega \left(\frac{1}{2}\lambda|Du|^2 - \lambda\nu^2|u|^2\right)dx \text{ (by (32.187))} \\
&\geq \left(\frac{\lambda}{2}C^{-1} - \lambda\nu^2\right)\int_\Omega |u|^2\,dx \text{ (by Poincaré's inequality).}
\end{aligned} \tag{32.188}$$

Hence we may define

$$\lambda_1 = \inf_H J. \tag{32.189}$$

We claim now that λ_1 is the minimum eigenvalue of L on H; that is, there exists a non-trivial $u \in H$ such that $Lu + \lambda_1 u = 0$ and λ_1 is the smallest number for which this is possible. To show this we choose a minimizing sequence $\{u_m\} \subset H$ such that $\|u_m\|_{L^2} = 1$ and $J(u_m) \to \lambda_1$.