29 Shiffman's Theorems

Recall that we defined a CBA as a minimal annulus $A \in S(-1,1)$ such that $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are continuous convex Jordan curves. In the article [76] published in 1956, Max Shiffman proved three elegant theorems about a CBA. They are as follows:

Theorem 29.1 *If A is a CBA, then $A \cap P_t$ is a strictly convex Jordan curve for every $-1 < t < 1$. In particular, $X : A_R \hookrightarrow S(-1,1)$ is an embedding.*

Theorem 29.2 *If A is a CBA and $\Gamma = \partial A$ is a union of circles, then $A \cap P_t$ is a circle for every $-1 \le t \le 1$.*

Theorem 29.3 *If A is a CBA and $\Gamma = \partial A$ is symmetric with respect to a plane perpendicular to xy-plane, then A is symmetric with respect to the same plane.*

We are going to prove the three Shiffman's theorems by means of the Enneper-Weierstrass representation. We have already proved a weaker version of Theorem 29.1, namely Theorem 27.2

Let us first prove Theorem 29.1. We follow the proof of Shiffman. We will write the immersion as $X = (x, y, z)$. For any $\zeta = re^{i\theta} \in A_R$, since X is conformal, by (27.124) we have

$$x_\theta^2 + y_\theta^2 = r^2(x_r^2 + y_r^2) + \frac{1}{(\log R)^2}.$$

The immersion $X : A_R \hookrightarrow S(-1,1)$ satisfies

$$x_\theta^2 + y_\theta^2 \ge \frac{1}{(\log R)^2}. \tag{29.150}$$

Since X is continuous on A_R, $A(1)$ and $A(-1)$ are convex and hence rectifiable. Moreover, $x(R,\theta)$ and $y(R,\theta)$ are functions of bounded variation. Thus $x_\theta(R,\theta)$ and $y_\theta(R,\theta)$ exist almost everywhere. Let I denote the set on which $x_\theta(R,\theta)$ and $y_\theta(R,\theta)$ both exist. We will first prove that:

Lemma 29.4 *For any $\theta \in I$,*

$$\lim_{r\to R} x_\theta(r,\theta) = x_\theta(R,\theta), \quad \lim_{r\to R} y_\theta(r,\theta) = y_\theta(R,\theta), \tag{29.151}$$

and

$$x_\theta^2(R,\theta) + y_\theta^2(R,\theta) \ge \frac{1}{(\log R)^2}. \tag{29.152}$$