28 The Existence of Minimal Annuli in a Slab

Given two Jordan curves Γ_1, Γ_2 in $\mathbf{R}^3$, does $\Gamma := \Gamma_1 \bigcup \Gamma_2$ bound a minimal annulus? This is called the Douglas-Plateau problem which is a generalisation of the original Plateau problem. If the answer to the Douglas-Plateau problem for a given Γ is yes, then we can ask that how many such minimal annuli are there?

These are very hard and interesting problems. Generally, they are attacked with concepts and techniques, such as those from the geometric measure theory which are quite different from the classical setting as in our notes,

One classical result due to Douglas says that if A_1 and A_2 are the areas of least area minimal disks bounded by Γ_1 and Γ_2 respectively, and

$$\inf\{\mathbf{Area}(S)\} < A_1 + A_2,$$

then there is a minimal annulus bounded by Γ. Here the infimum is taken over all surfaces of annular type bounded by Γ. See [13], or [9].

In many cases the answers to the Douglas-Plateau problem are no. One example is that of two coaxial unit circles C_1 and C_2. If the distance d between their centres is large then $C_1 \cup C_2$ cannot bound a catenoid, and therefore as Shiffman's second theorem (Theorem 29.2) shows, $C_1 \cup C_2$ cannot bound a minimal annulus.

When Γ_1 and Γ_2 are smooth convex planar Jordan curves lying in parallel (but different) planes, the Douglas-Plateau problem has a very satisfactory answer. The combined result of Hoffman and Meeks [28], and Meeks and White [53], says,

Let $\Gamma = \Gamma_1 \bigcup \Gamma_2$. Then there are exactly three cases:

1. *There are exactly two minimal annuli bounded by Γ, one is stable and one is unstable.*

2. *There is a unique minimal annulus A bounded by Γ; it is almost stable in the sense that the first eigenvalue of L_A is zero. This case is not generic.*

3. *There are no minimal annuli bounded by Γ.*

4. *Moreover, if A is a minimal annulus bounded by Γ, then the symmetry group of A is the same as the symmetry group of Γ.*

We are not going to discuss the Douglas-Plateau problem in these notes. Rather, we would like to point out some necessary conditions on Γ if it bounds a minimal annulus.

The next theorem is due to Osserman and Schiffer [70], we follow their proof.

Theorem 28.1 *Let δ_1, δ_2, c, d be positive numbers satisfying*

$$\left(\frac{c^2}{2} + d^2\right)^{1/2} \geq \delta_1 + \delta_2. \tag{28.130}$$