27 Minimal Annuli in a Slab

Recall that a catenoid is a rotation surface, hence is foliated by circles in parallel planes. A good question to ask is whether there are other minimal annuli which can be foliated by circles. It was B. Riemann [72] and Enneper [14] who solved this problem very satisfactorily. The answer is that there is only one one-parameter family of such surfaces up to a homothety. Each minimal annuli in this one-parameter family is contained in a slab and foliated by circles, and its boundary is a pair of parallel straight lines. Rotating repeatly about these boundary straight lines gives a one-parameter family of singly periodic minimal surface; these surfaces are called *Riemann's examples.*

For the details of the proof of existence and other properties of Riemann's examples, see [61], section 5.4, Cyclic minimal surfaces. For constructions of Riemann's examples using the Weierstrass functions please see [25]. It is also known that a pair of parallel straight lines can only bound a piece of Riemann's example, if they bound any minimal annulus at all, see for example, [17].

Now we are going to study minimal annuli in a slab. Let $P_t = \{(x, y, z) \in \mathbf{R}^3 | z = t\}$ and $S(t_1, t_2) = \{(x, y, z) \in \mathbf{R}^3 | t_1 \le z \le t_2,\ t_1 < t_2\}$. Consider a minimal annulus $X : A_R \hookrightarrow S(t_1, t_2)$ such that $X(\{|z| = 1/R\}) \subset P_{t_1}$, $X(\{|z| = R\}) \subset P_{t_2}$ and X is continuous on A_R. We will call such a minimal annulus a *minimal annulus in a slab*. By a homothety we can normalize t_1 and t_2 such that $t_1 = -1$ and $t_2 = 1$. We will denote the image $X(A_R) \subset S(-1, 1)$ by A and let $A(t) = A \cap P_t$ for $-1 \le t \le 1$. When discussing a minimal annulus in a slab, we often just refer to it by the image $A = X(A_R)$.

We want to derive the Enneper-Weierstrass representation of a minimal annulus in a slab. Let A be a minimal annulus in a slab. The third coordinate function X^3 is harmonic, $X^3|_{\{|z|=1/R\}} = -1$, and $X^3|_{\{|z|=R\}} = 1$. By uniqueness of solutions to the Dirichlet problem

$$\begin{cases} \triangle u = 0 & \text{in } \mathrm{Int}(A_R) \\ \\ u|_{\{|z|=1/R\}} = -1, & u|_{\{|z|=R\}} = 1, \end{cases}$$

where $\mathrm{Int}(A_R)$ is the interior of A_R, we have $X^3 = \frac{1}{\log R} \log |z|$, and

$$\omega_3 = f(z)g(z)dz = 2\frac{\partial}{\partial z} X^3\, dz = \frac{d}{dz}\left(\frac{1}{\log R} \log z\right) dz = \frac{1}{\log R}\frac{1}{z}\, dz.$$

Hence $f(z) = \frac{1}{\log R}\frac{1}{zg(z)}$. Here of course g is the Gauss map in the Enneper-Weierstrass representation and $f(z)dz = \eta$. Thus by (6.26) we have

$$\begin{cases} \omega_1 &=& \frac{1}{\log R}\frac{1}{2z}(\frac{1}{g} - g)dz \\ \\ \omega_2 &=& \frac{1}{\log R}\frac{i}{2z}(\frac{1}{g} + g)dz \\ \\ \omega_3 &=& \frac{1}{\log R}\frac{1}{z}dz. \end{cases} \tag{27.124}$$