25 Minimal Annuli

The catenoid is topologically an annulus, and is the only embedded complete minimal annulus of finite total curvature by Theorem 18.1. Since any complete minimal surface has annular end, we want to study minimal surfaces of annular type, with or without boundary.

All the results in this section are due to Osserman and Schiffer [70].

First we fix $A := \{r_1 < |z| < r_2\} \subset \mathbf{C}$, $0 < r_1 \leq 1 \leq r_2 \leq \infty$ (by Lemma 9.1 and Proposition 9.2 we can always select such a representation of A). Let $X : A \to \mathbf{R}^3$ be a minimal annulus. Let g and $\eta = f(z)dz$ be the Weierstrass data for X and ϕ_i be as (6.15), $i = 1, 2, 3$. Let $\psi_i = z\phi_i$. Write $X = (X_1, X_2, X_3)$ and let

$$t = \log r = \log |z|.$$

We define

$$\overline{X}(r) := \frac{1}{2\pi}\int_0^{2\pi} X(re^{i\theta})d\theta.$$

Lemma 25.1

$$\frac{d^2\overline{X}(r)}{dt^2} = 0.$$

Proof.

$$\begin{aligned}
\frac{d^2\overline{X}(r)}{dt^2} &= r\frac{d}{dr}\left(r\frac{d\overline{X}(r)}{dr}\right) = \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}X(r^{i\theta})\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} r^2 \triangle X \, d\theta = 0,
\end{aligned}$$

since

$$0 = \triangle X = r^{-1}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}X(re^{i\theta})\right] + r^{-2}\frac{\partial^2 X(re^{i\theta})}{\partial\theta^2}$$

and

$$\int_0^{2\pi} \frac{\partial^2 X(re^{i\theta})}{\partial\theta^2} \, d\theta = 0.$$

□

Let $C = \{|z| = 1\} \subset A$. Since C is the generator of the first homology group of A, by (17.72), we can define

$$\begin{aligned}
\mathbf{Flux}(X) &= \mathbf{Flux}(C) = \Im \int_C \phi(z)dz = \Im \int_{|z|=r} \phi(z)dz = -i\int_{|z|=r} \phi(z)dz \\
&= \int_0^{2\pi} \phi(re^{i\theta})re^{i\theta}d\theta = \int_0^{2\pi} \psi(re^{i\theta})d\theta \qquad (25.91)
\end{aligned}$$