24 Complete Minimal Surfaces of Finite Topology

Based on Corollary 24.5, Hoffman and Meeks made the following conjecture in [31]:

Conjecture 24.1 *Let $X : M \hookrightarrow \mathbf{R}^3$ be a properly embedded complete minimal surface of finite topology with more than one end. Then X has finite total curvature.*

With the help of Theorem 23.1, we can give a clearer picture of properly embedded complete minimal surfaces with more than one end.

Theorem 24.2 *Suppose M is a properly embedded minimal surface in $\mathbf{R}^3$ that has two annular ends, each having infinite total curvature. Then these two ends have representatives E_1 , E_2 satisfying the following:*

1. *There exist disjoint closed halfspaces $\mathbf{H}_1$, $\mathbf{H}_2$ such that $E_1 \subset \mathbf{H}_1$ and $E_2 \subset \mathbf{H}_2$.*

2. *All other annular ends of M are asymptotic to flat planes parallel to $\partial\mathbf{H}_1$.*

3. *M has only a finite number of normal vectors parallel to the normal vector of $\partial\mathbf{H}_1$.*

Proof. Given two properly embedded minimal annuli A_1, A_2 each with compact boundary curve, if $A_1 \cap A_2 = \emptyset$ then there exists a standard barrier between them. This means that there exists a half-catenoid or a plane C such that outside of a sufficiently large ball B the barrier C is disjoint from $A_1 \cup A_2$ and also $C \cup B$ separates $A_1 - B$ from $A_2 - B$. Now consider the two annular ends E_1 and E_2 of M with infinite total curvature; Theorem 23.1 implies that C must be a plane. Since C is disjoint from $E_1 \cup E_2$ outside of some ball, $C \cap (E_1 \cup E_2)$ is compact. Hence, after removing compact subannuli of E_1 and E_2, we may choose E_1 and E_2 to lie in the disjoint halfspaces determined by C. The weak maximum principle at infinity (Remark 15.3) implies that E_i and C stay a bounded distance apart for $i = 1, 2$. Therefore, the distance from C to $E_1 \cup E_2$ is greater than some $\epsilon > 0$. It follows that we can choose closed disjoint halfspaces $\mathbf{H}_1$, $\mathbf{H}_2$ with $E_1 \subset \mathbf{H}_1$ and $E_2 \subset \mathbf{H}_2$. This proves the first statement of the theorem.

Suppose now that E_3 is another annular end of M that is disjoint from E_1 and E_2. Corollary 22.6 says that at least one of E_1, E_2 and E_3 lying between two standard barriers. By Proposition 22.3, an end lies between two standard barriers must have finite total curvature. Hence it is evident that E_3 has finite total curvature and lies between two standard barriers, and hence between E_1 and E_2. If E_3 is a catenoid end, then either E_1 or E_2 lies above a catenoid. By Theorem 23.1, E_1 or E_2 has finite total curvature, contradicting our hypotheses. Hence E_3 is asymptotic to a flat plane P. By the weak maximum principle at infinity the end of this plane P stays a positive distance from both E_1 and E_2. This implies that P intersects both E_1 and E_2 in a compact set and hence E_1 and E_2 have proper subends that are a positive distance from P. Hence we may assume that $E_i \cap P = \emptyset$ for $i = 1, 2$. By Theorem 16.1, the convex hulls of