22 Standard Barriers and The Annular End Theorem

The study of ends of complete minimal surfaces leads to the Annular End Theorem of Hoffman and Meeks [29], and its corollaries.

Theorem 22.1 (The Annular End Theorem) *If M is a properly embedded minimal surface in $\mathbf{R}^3$, then at most two distinct annular ends of M can have infinite total curvature.*

To prove the Annular End Theorem, we need some preparation. First we introduce the notion of a *standard barrier.*

Definition 22.2 A *standard barrier* in $\mathbf{R}^3$ is one of the following two minimal surfaces with boundary: the complement of a disk in a plane in $\mathbf{R}^3$; a component of the complement of a simple, closed, homotopically nontrivial curve on a catenoid.

We will say that a surface $M \subset \mathbf{R}^3$ *admits a standard barrier* if it is disjoint from some standard barrier. We will use the word "eventually" to mean "outside of some sufficiently large compact set of $\mathbf{R}^3$". Thus, two surfaces $M \subset \mathbf{R}^3$ and $N \subset \mathbf{R}^3$ are "eventually disjoint" if they have compact intersection. It is straightforward to see that M admits a standard barrier if and only if it is eventually disjoint from some standard barrier.

Given a standard barrier S and a ball B large enough to contain ∂S, it is clear that $S - B$ divides $\mathbf{R}^3 - B$ into two components. Two surfaces M, $N \subset \mathbf{R}^3$ will be said to be *separated by a standard barrier* if such an S and B can be found so that M and N eventually lie in different components of $\mathbf{R}^3 - (B \cup S)$.

Two disjoint standard barriers divide the complement of a sufficiently large ball $B \subset \mathbf{R}^3$ into three components, only one of which contains portions of both barriers on its boundary. A surface $M \subset \mathbf{R}^3$ that eventually lies in such a component will be said to *lie between two standard barriers.* After a rotation of $\mathbf{R}^3$, if necessary, the region of $\mathbf{R}^3$ between two standard barriers eventually lies in the complement of any $X_c = \{x_1^2 + x_2^2 = (x_3/c)^2\}$ (in the component that contains $P^0 - \{0\}$) for any $c > 0$, no matter how small. It follows from Theorem 21.1 and Remark 21.4 that:

Proposition 22.3 *If $X : M \hookrightarrow \mathbf{R}^3$ is a properly immersed complete minimal surface of finite topology, with compact boundary ∂M, and eventually lies between two standard barriers, then M must have finite total curvature.*

Our strategy in proving the Annular End Theorem is to trap ends between standard barriers. The next lemma contains the critical technical construction.

Before proving the lemma, we introduce the notion of *linking number.*