20 The Second Variation and Stability

We now introduce the concept of *stability* of minimal surfaces which will play an important role in the proof of several theorems in the remainder of these notes.

Let Ω be a precompact domain in a Riemann surface M, $X : \Omega \to \mathbf{R}^3$ a minimal surface. From the calculus of variations definition of a minimal surface, we know that X is a minimal surface if and only if the area A of X is a stationary point of the area functional $A(t)$ for any variation $X(t)$. Note that being stationary does not mean that X has minimum area among all surfaces with the same boundary.

To study when X has locally minimum area, naturally we study the second variation, namely the second derivative $A''(0)$ of the area functional for any variation family $X(t)$. From calculus we know that if $A''(0) > 0$ then $A(0)$ is a local minimum. Note that the word *local* is significant, there are minimal surfaces such that $A''(0) > 0$ for any variation family, yet those surfaces do not have minimum area. Hence we define that X is *stable* if $A''(0) > 0$ for all possible variation families $X(t)$, otherwise X is *unstable*. Sometimes one says X is *almost stable* if $A''(0) \geq 0$.

It is important to express the formula for the second variation of X via the geometric quantities of X. Let (u^1, u^2) be the local coordinates of Ω. We use the fact that X is conformal harmonic, and write $\Lambda^2 = |X_1|^2 = |X_2|^2$, $\triangle = D_{11} + D_{22}$.

From (3.4),

$$\frac{dA(t)}{dt} = -2\int_\Omega H(t)(E(t)\bullet N(t))\,dA_t,$$

where $E(t) = \partial X(t)/\partial t$, $H(t)$ is the mean curvature of $X(t)$, and $N(t)$ is the Gauss map of $X(t)$. Let $E = \alpha X_1 + \beta X_2 + \gamma N$. Since $H(0) = 0$ we have

$$\frac{d^2A(t)}{dt^2}\Big|_{t=0} = -2\int_\Omega \frac{dH(t)}{dt}\Big|_{t=0}(E\bullet N)\,dA_0,$$

where we write $E = E(0)$, etc. Now suppose that each $X(t)$ is a C^2 surface, and the first and second fundamental forms are given on an isothermal coordinate chart U by

$$g_{ij}(t) = X_i(t)\bullet X_j(t), \quad (g^{ij}(t)) = (g_{ij}(t))^{-1}, \quad h_{ij}(t) = X_{ij}(t)\bullet N(t).$$

Then

$$H(t) = \frac{1}{2}\sum_{i,j} g^{ij}(t)h_{ij}(t),$$

hence

$$\frac{dH(t)}{dt}\Big|_{t=0} = \frac{1}{2}\sum_{i,j}\frac{dg^{ij}(t)}{dt}\Big|_{t=0}h_{ij} + \frac{1}{2}\sum_{i,j} g^{ij}\frac{dh_{ij}(t)}{dt}\Big|_{t=0},$$

where we write $g^{ij}(0) = g^{ij}$, etc. From

$$\sum_j g^{ij}(t)g_{jk}(t) = \delta_{ik}, \quad g^{ij} = \Lambda^{-2}\delta_{ij},$$