19 The Gauss Map of Complete Minimal Surfaces

Let $X : M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Let g and η be the Weierstrass data for X. The question in this section is how many points does the set $\mathbf{C} \cup \{\infty\} - g(M)$ have? We will only prove a relatively easy theorem due to Osserman, which will be useful when we discuss the behavior of minimal annuli. At the end of this section we will give an up to date survey of partial results for this problem.

To prove the theorem of Osserman mentioned above we have to introduce the concept of *capacity*. Since we only need describe when a set has capacity zero, we will only define zero capacity sets.

Definition 19.1 Let $D \subset \mathbf{C}$ be a closed set. Then D has *capacity zero* if and only if the function $\log(1+|z|^2)$ has no harmonic majorant in $\mathbf{C} - D$, i.e, there is no harmonic function $h : \mathbf{C} - D \to \mathbf{R}$ such that

$$\log(1+|z|^2) \leq h(z), \quad z \in \mathbf{C} - D.$$

Note that any finite set in $\mathbf{C}$ has capacity zero.

Theorem 19.2 *Let $X : M_r(:= \{1 \leq |z| < r \leq \infty\}) \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Then either the Gauss map g tends to a single limit as $|z| \to r$, or else in each neighbourhood of $\{|z| = r\}$ g takes on all points of $\mathbf{C} \cup \{\infty\}$ except for at most a set of capacity zero.*

Proof. Let $\eta = f(z)dz$. Suppose now that in some neighbourhood of $\{|z| = r\}$ g omits a set Z of positive capacity. This means that for some $1 \leq r_1 < r$, the function $w = g(z)$ omits Z in the domain $D' := \{r_1 < |z| < r\}$. Hence there exists a harmonic function $h(w)$ defined in $\mathbf{C} - Z \supset g(D')$ such that $\log(1+|w|^2) \leq h(w)$. Since the induced metric by X on M_r is $\Lambda^2 = \frac{1}{4}|f|^2(1+|g|^2)^2$, we have

$$\log \Lambda(z) \leq \log \frac{|f|}{2} + h(g(z)).$$

Since g and f are holomorphic, the right hand side of the above inequality is harmonic. By Lemma 10.5 and Proposition 10.6, $r = \infty$. But then g could not have an essential singularity at infinity by Picard's theorem. Thus g tends to a limit, finite or infinite, as z tends to infinity. □

Remark 19.3 Once we know that $r = \infty$ and g has a limit at infinity, we know that X has finite total curvature. The argument is as follows:

By a rotation if necessary we may assume that g has a pole at ∞. Then $g(z) = z^n h(z)$ where $h(\infty) \neq 0$ and $n > 0$. Since

$$\frac{4|g'|^2}{(1+|g|^2)^2} = O(|z|^{-2n}) \quad \text{at } \infty,$$