18 Uniqueness of the Catenoid

Let $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be a complete minimal surface, where S_k is a closed Riemann surface of genus k. We say that X has genus k.

The catenoid has genus zero by this definition. We have already proved that the catenoid is the only embedded complete minimal surface of total curvature -4π and is the only minimal surface which is a rotation surface. Schoen [74] proved that the catenoid is the only complete minimal surface with exactly two annular ends and finite total curvature. Thus the catenoid has many special features which describe it uniquely.

In 1989, López and Ros proved the following remarkable theorem [49].

Theorem 18.1 *The catenoid is the only embedded genus zero non-planar minimal surface of finite total curvature.*

The proof of Theorem 18.1 is a combination of the flux formula and the maximum principle at infinity. We will give a proof here adapted from [71].

Another key ingredient in the proof of Theorem 18.1 is deformation. Suppose that $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is a minimal surface. If for any loop $\Gamma \subset M$, $\mathbf{Flux}(\Gamma)$ is a vertical vector, i.e., parallel to $(0, 0, 1)$, then we say that X has *vertical flux.* By Proposition 17.1, we see that X has vertical flux if and only if for any loop Γ,

$$\int_\Gamma \eta = 0, \quad \text{and} \quad \int_\Gamma g^2\eta = 0, \tag{18.81}$$

where g and η are the Weierstrass data for X.

Let $\lambda \in (0, \infty)$ and $\eta_\lambda = \lambda^{-1}\eta$, $g_\lambda = \lambda g$. Consider the corresponding Enneper-Weierstrass representation,

$$\omega_1^\lambda = \frac{1}{2}(\lambda^{-1} - \lambda g^2)\eta; \quad \omega_2^\lambda = \frac{i}{2}(\lambda^{-1} + \lambda g^2)\eta; \quad \omega_3^\lambda = g\eta = \omega_3.$$

If X has vertical flux, then we have a family of well defined minimal surfaces, deformations of the original surface, given by

$$X^\lambda = \Re \int (\omega_1^\lambda, \omega_2^\lambda, \omega_3^\lambda). \tag{18.82}$$

Note that the third coordinate function of X^λ does not depend on λ.

A point $p \in M$ such that $g(p) = 0$ or ∞ is called a *vertical point* of X. Since $g_\lambda = \lambda g$, if p is a vertical point of X then p is also a vertical point of X^λ, and vice versa. We first investigate the behaviour of X^λ when p is a vertical point.

Lemma 18.2 *Suppose that X has vertical flux and is non-planar. If p is a vertical point, then X^λ is not an embedding when λ is sufficiently large or sufficiently small.*