16 The Convex Hull of a Minimal Surface

Recall that the *convex hull* $H(E)$ of a set $E \subset \mathbf{R}^n$ is defined as

$$H(E) = \bigcap_{E \subset \mathbf{H}} \mathbf{H}$$

where $\mathbf{H}$ is a halfspace in $\mathbf{R}^n$. Of course, if E is not contained in any halfspace, then $H(E) = \mathbf{R}^n = \bigcap_{\emptyset} \mathbf{H}$.

We want to study the convex hull of a minimal surface.

Let M be conformally a bounded plane domain and $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface such that X is continuous on $\overline{M}$. If $\partial M \neq \emptyset$, then a simple application of the maximum principle for harmonic functions shows that $X(M) \subset H(X(\partial M))$, where $H(X(\partial M))$ is the convex hull of $X(\partial M)$.

Exercise : Prove this fact.

Now using the the Halfspace Theorem, we can prove more.

Theorem 16.1 ([32]) *Suppose that $M \subset \mathbf{R}^3$ is a proper, complete, connected minimal surface in $\mathbf{R}^3$, whose boundary ∂M, which may be empty, is a compact set. Then exactly one of the following holds:*

1. $H(M) = \mathbf{R}^3$;
2. $H(M)$ *is a halfspace;*
3. $H(M)$ *is a closed slab between two parallel planes;*
4. $H(M)$ *is a plane;*
5. $H(M)$ *is a compact convex set. This case occurs precisely when M is compact.*

Furthermore, ∂M has nonempty intersection with each boundary component of $H(M)$.

Remark 16.2 We note that all of these cases are possible. For 1 and 2, examples are the catenoid and half-catenoid. For 3 we could take any of the examples in theorem 14.8 and consider the portion of these surfaces in the slab $|x_3| \leq 1$. This surface is bounded by two Jordan curves. For 4 we have a plane and 5 is the case for any compact example.

Proof of Theorem 16.1. Suppose now that cases 1, 4 and 5 do not occur. To prove that case 2 or case 3 must occur we need show that if H_1 and H_2 are distinct smallest halfspaces containing M, then $P_1 = \partial H_1$ and $P_2 = \partial H_2$ are parallel planes. Suppose now that P_1 and P_2 are not parallel planes. We shall derive a contradiction.

The interior of M cannot have a point in common with $P_1 \cup P_2$. (If it did then the maximum principle for minimal surface (see Theorem 4.4 and Remark 4.6) implies it