15 The Halfspace Theorem and The Maximum Principle at Infinity

By Jorge and Meeks' theorem, we know that if we stand at infinity to view a complete minimal surface of finite total curvature, it looks like several planes passing through origin,

We will further discuss the image of such a surface. The basic theorem in this section is the Halfspace Theorem due to Hoffman and Meeks [32], its proof is surprisingly simple.

Theorem 15.1 (Halfspace Theorem) *A connected, proper, possibly branched, nonplanar complete minimal surface M in $\mathbf{R}^3$ is not contained in a halfspace.*

Proof. Suppose the theorem is false.

Define $\mathbf{H}_t := \{(x_1, x_2, x_3) \,\big|\, x_3 \geq t\}$, $P_t = \partial \mathbf{H}_t$, $t \in \mathbf{R}$. By a translation and rotation, we may assume that $M \subset \mathbf{H}_0$. Let $T := \sup\{t \,\big|\, M \subset \mathbf{H}_t\}$. If $p \in M \cap P_T$, then P_T is the tangent plane T_pM. By Corollary 4.5, M must be on both sides of P_T, contradicting the fact that $M \subset \mathbf{H}_{T-\epsilon}$, any $\epsilon > 0$. Hence $M \cap P_T = \emptyset$. By a translation, we may assume that $T = 0$.

Let M_ϵ be the downward translation of M, then $M_\epsilon \cap P_0 \neq \emptyset$ for any $\epsilon > 0$. Let $C = C_1$ be the half-catenoid $\{(x_1, x_2, x_3) \,\big|\, x_1^2 + x_2^2 = \cosh^2(x_3),\ x_3 < 0\}$. By choosing $\epsilon > 0$ small enough, we may insure that $M_\epsilon \cap C_1 = \emptyset$ and $M_\epsilon \cap D_1 = \emptyset$, where D_1 is the unit-disk in P_0. Specifically, let $d > 0$ be the distance from M to the disk of radius $R = \cosh(1) > 1$. Outside the cylinder over D_R, C_1 lies below the plane P_{-1}. We will choose $\epsilon < \frac{1}{2}\min\{1, d\}$ small enough so that $M_\epsilon \cap C_1 = \emptyset$ and $M_\epsilon \cap P_0 \neq \emptyset$.

Denote by C_t the homothetic shrinking of C_1 by t, $0 < t \leq 1$. Observing that C_t converges smoothly, away from 0, to $P_0 - \{0\}$ we may conclude from the previous paragraph that $C_t \cap M_\epsilon \neq \emptyset$ for t sufficiently small, that $C_t \cap M_\epsilon$ lies outside the cylinder over D_1 for all t, and that $C_t \cap M_\epsilon = \emptyset$ for t close to 1.

Let $S = \{t \,|\, C_t \cap M_\epsilon \neq \emptyset\}$ and $T = \text{lubS}$. We claim that $T \in S$, i.e., $C_T \cap M_\epsilon \neq \emptyset$, thus $T < 1$.

If T is an isolated point of S, we are done. If not, we can find an increasing sequence $t_n \to T$, with $t_0 > T/2$, such that there exist points $p_n \in C_1$ with $t_n p_n \in C_{t_n} \cap M_\epsilon$. If $p_n = (x_n, y_n, z_n)$, we must have $t_n z_n \geq -\epsilon$ which implies $z_n \geq -\epsilon/t_n \geq -2\epsilon/T$. This means that p_n lies on the bounded closed subset $X_T := \{(x_1, x_2, x_3) \in C_1 \,\big|\, x_3 \geq -2\epsilon/T\}$ and must therefore possess a convergent subsequence. If $\{p_j\}$ is that subsequence and $p_j \to p_0 \in C_1$, then $t_j p_j \in C_{t_j} \cap M_\epsilon$. Since X_T is compact and M is proper, $\{t_j p_j\}$ must have a convergent subsequence in M_ϵ, still denoted by $\{t_j p_j\}$, and by continuity, $Tp_0 \in C_T \cap M_\epsilon$. This proves that $C_T \cap M_\epsilon \neq \emptyset$.

Since the boundary of C_T lies inside $D_1 \subset P_0$, and that disk is disjoint from M_ϵ, Tp_0 must be an interior point of C_T. Moreover, the fact that $T < 1$ and $C_t \cap M_\epsilon = \emptyset$ for $t > T$ means that C_T meets M_ϵ at Tp_0, but lies locally on one side of M_ϵ near