14 Examples of Complete Minimal Surfaces

We have discussed complete minimal surfaces without a single example. This section is designed to give some examples and their Enneper-Weierstrass representations. But first let us sum up what we have known, in order to simplify the following discussion.

Suppose now that $X : S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$ is a complete minimal immersion of finite total curvature. Then by the Enneper-Weierstrass representation, X is given by

$$X = \Re \int \left(\frac{1}{2}(1 - g^2)\eta, \frac{i}{2}(1 + g^2)\eta, g\eta \right) + C.$$

If we want X to be embedded, then g and η should satisfy certain conditions. Here is an important necessary condition.

Theorem 14.1 *Let S be a complete minimal immersion in $\mathbf{R}^3$ of finite total curvature, defined by $X : M \to \mathbf{R}^3$, where $M \cong S_k - \{p_1, \cdots, p_r\}$. Let g and the holomorphic 1-form η be the Enneper-Weierstrass data for X. Then $\eta \neq 0$ at a point $p \in M$ unless g has a pole at p, and if g has a pole at $p \in M$ of order m, then η has a zero of order $2m$ at p.*

Suppose that E_i is an embedded end corresponding to p_i. If g has a pole of order $k \geq 1$ at p_i, then η has a zero of order $2k - 2$ at p_i. If g takes on a finite value at p_i, then η has a pole of order 2 at p_i. Furthermore, p_i is a branch point of g if and only if E_i is a flat end.

Proof. From $\Lambda^2 = \frac{1}{4}|f|^2(1 + |g|^2)^2$, we see that to make $0 < \Lambda < \infty$ on M, the zeros and poles of g and η must be as stated in the theorem.

We have already seen that an end is embedded if and only if Λ has order 2, thus g and η has to satisfy the conditions stated in this theorem.

The last statement is Remark 11.13. □

Now let us see some examples. The first one, the *catenoid*, is quite a classical one, it was discovered in 1741 by Euler, see [61], page 5. By the way, one can find a very interesting history of minimal surfaces in [61].

Example 14.2 (Catenoid) *Let $M = \mathbf{C} - \{0\}$, $g(w) = w$, $\eta = \frac{dw}{w^2}$. The Enneper-Weierstrass representation of the catenoid is given by the three 1-forms:*

$$\omega_1 = \frac{1}{2}\frac{1}{w^2}(1 - w^2)dw, \quad \omega_2 = \frac{i}{2}\frac{1}{w^2}(1 + w^2)dw, \quad \omega_3 = \frac{1}{w}dw.$$

The total curvature of the catenoid is -4π since $g(w) = w$ has degree 1. It has genus zero and two ends. At 0 and ∞ we see that ϕ_1 and ϕ_2 have poles of order 2 and ϕ_3 has a pole of order 1, so the two ends are embedded and they are catenoid ends since $g(w) = w$ has no branch points. We can prove that the catenoid is embedded and is a rotation surface.