13 Total Curvature of Branched Complete Minimal Surfaces

Let $X: M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface with finite total curvature. Osserman's theorem says that conformally $M = S_k - \{p_1, \cdots, p_n\}$, $n \geq 1$, where S_k is a closed Riemann surface of genus k. Each p_i corresponds to an end E_i of M. Using Theorem 12.1, we can prove:

Theorem 13.1 *The total curvature of X is*

$$K(M) = 2\pi \left(\chi(M) - \sum_{i=1}^{n} I_i \right), \tag{13.57}$$

where $\chi(M) = 2(1-k) - n$ is the Euler characteristic of M and I_i is the multiplicity of E_i.

Proof. Let $\Gamma_i^r = X^{-1}(rW_i^r)$ be as in the proof of Theorem 12.1. Let $p_i \in D_i^r$ be the disk in S_k such that $\partial D_i^r = \Gamma_i^r$. When r is large enough the D_i^r's are disjoint from each other. Then $M_r := S_k - \bigcup_{i=1}^n \partial D_i^r$ is a Riemann surface with boundary $\bigcup_{i=1}^n \partial D_i^r$ and $\chi(M_r) = \chi(M)$. Now by the Gauss-Bonnet formula we have

$$\int_{M_r} K dA + \sum_{i=1}^{n} \int_{\Gamma_i^r} \kappa_g \, ds = 2\pi\chi(M_r) = 2\pi\chi(M),$$

where κ_g is the geodesic curvature. Since $W_i^r = \frac{1}{r}X(\Gamma_i^r)$ converges in the C^∞ sense to a great circle on S^2 with multiplicity I_i and X is an isometric immersion, we have

$$\lim_{r \to \infty} \int_{\Gamma_i^r} \kappa_g \, ds = 2\pi I_i.$$

Taking limit we have

$$K(M) = \int_M K dA = 2\pi \left(\chi(M) - \sum_{i=1}^{n} I_i \right). \tag{13.58}$$

□

In the remainder of this section, our surfaces will be branched minimal surfaces. Note that the concepts of completeness, properness, etc., can be easily generalised to branched minimal surfaces.

The Enneper-Weierstrass representation of a branched complete minimal surface of finite total curvature $X: M \to \mathbf{R}^3$ is given by

$$X(p) = \Re \int_{p_0}^{p} \left(\frac{1}{2}(1-g^2), \ \frac{i}{2}(1+g^2), \ g \right) \eta + C, \tag{13.59}$$