12 Complete Minimal Surfaces of Finite Total Curvature

To have a better understanding of a complete immersed minimal surface of finite total curvature, we will prove a theorem due to Jorge and Meeks which says that if one looks at the surface from infinity, then the surface looks like a finite number of planes passing through the origin.

Let $X : M \cong S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be an immersed complete surface. Let $S^2(r)$ be the sphere centred at $(0,0,0)$ with radius r. Let $Y_r = X(M) \cap S^2(r)$ and

$$W_r = \frac{1}{r} Y_r \subset S^2.$$

Theorem 12.1 ([38]) *Suppose that the Gauss map on M extends continuously to S_k. Then*

1. *$X : M \cong S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is proper.*
2. *For large r, $W_r = \{\gamma_1^r, \cdots, \gamma_n^r\}$ consists of n immersed closed curves on S^2.*
3. *γ_i^r converges in the C^1 sense to a geodesic of S^2 with multiplicity $I_i \geq 1$ as r goes to infinity.*
4. *If X is a minimal surface then the convergence in 3 is C^∞.*
5. *X is embedded at an end corresponding to p_i if and only if $I_i = 1$.*

Proof. We need only consider a neighbourhood of a puncture p. Let $D^* = D - \{p\}$ be a punctured disk and ∂D be compact. Suppose that

$$N = \lim_{|z| \to 0} N(z),$$

and that

$$N \bullet N(z) = \cos\theta \geq \frac{\sqrt{3}}{2} \text{ for } 0 \leq \theta \leq \frac{\pi}{6} \tag{12.52}$$

for all $z \in D^*$. Let π be a plane containing the line generated by N and let $\Gamma = X^{-1}(\pi)$. Since $N \bullet N(z) \geq \sqrt{3}/2$, X is transversal to π. It follows that Γ consists of points in ∂D and connected curves (in fact, the interior of $X^{-1}(\pi)$ is a one-dimensional manifold). Let γ be a connected component of Γ that is a curve.

We will consider coordinates (t, y) in π such that the y-axis is the line generated by N. It follows from (12.52) that the tangent vector of $X(\gamma)$ is never collinear with N. Thus $X(\gamma)$ is the graph of a function $y(t)$. The angle between the normal vector $(-y', 1)$ of $X(\gamma)$ and N is less than or equal to θ. Therefore

$$\frac{1}{\sqrt{1 + y'(t)^2}} \geq \cos\theta,$$