# 11 Ends of Complete Minimal Surfaces

By Osserman's theorem, any complete minimal surface of finite total curvature is an immersion $X : M = S_k - \{p_1, \cdots, p_r\} \hookrightarrow \mathbf{R}^3$, where $S_k$ is a closed Riemann surface of genus $k$. Consider conformal closed disks $D_i \subset S_k$ such that $p_i \in D_i$ and $p_j \notin D_i$ for $j \neq i$. Denote $D_i^* := D_i - \{p_i\}$. For any such $D_i$, the restriction $X : D_i^* \hookrightarrow \mathbf{R}^3$ is called a *representative of an end of* $X$ *at* $p_i$ or simply *an end.* When we say that some property holds at an end of $X$ at $p_i$, for example embeddedness, we mean that there is a disk like domain $D_i$ such that for any disk like domain $p_i \in U_i \subset D_i$, $X : U_i - \{p_i\}$ satisfies the property. Such a representative $X : U_i - \{p_i\} \to \mathbf{R}^3$ is called a *subend* of the end $X : D_i^* \hookrightarrow \mathbf{R}^3$.

Osserman's theorem says that the Gauss map $g$ extends to $p_i$ and the extended $g$ is a meromorphic function. Since $N = \tau^{-1} \circ g$ we have a well defined normal vector $N(p_i)$ at $p_i$, which we call the *limit normal* at $p_i$. This also defines a *limit tangent plane* at the end $E_i$ corresponding to $p_i$.

Intuitively, and we will prove it later (see Proposition 11.5), $E_i = X(D_i^*) \subset \mathbf{R}^3$ is an unbounded set. Moreover, since $M - \bigcup_{i=1}^r D_i^*$ is precompact, $X(M) - \bigcup_{i=1}^r E_i$ is bounded. Thus if $X$ is an embedding, an end $E_i$ is just a connected component of $X(M) - B$, where $B$ is any sufficiently large ball in $\mathbf{R}^3$ centred at 0.

In this section, all ends considered are ends of some complete minimal surface of finite total curvature.

Now consider the Enneper-Weierstrass representation of the complete minimal surface $X : M \hookrightarrow \mathbf{R}^3$. By (6.20)

$$\Lambda^2 = \frac{1}{2}\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right).$$

Now let $r : [0, 1) \to D_i^*$ be a regular curve such that $|r'(t)| = 1$ and $\lim_{t \to 1} r(t) = p_i$. By completeness,

$$\int_0^1 \Lambda(r(t))\, |r'(t)|\, dt = \infty.$$

This implies that $\Lambda(q) \to \infty$ as $q \to p$. Since $\phi_i$'s are meromorphic, one of them must have a pole at $p$. Hence let $z$ be the local coordinate of $D_i$ such that $z(p_i) = 0$, we must have

$$\Lambda^2 = \frac{1}{2}\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right) \sim \frac{c}{|z|^{2m}}, \tag{11.46}$$

where $c > 0$ and $m \geq 1$ is an integer.

**Definition 11.1** If $\Lambda^2 \sim c/|z|^{2m}$ at an end, we say that $\Lambda$ has order $m$ at that end.

**Remark 11.2** Since $\Lambda^2$ is the pull back metric of $X : M \to \mathbf{R}^3$, we see that the order of $\Lambda$ is invariant under an isometry in $\mathbf{R}^3$. Precisely, if $A$ is an isometry of $\mathbf{R}^3$ then $AX$ and $X$ has the same pull back metric $\Lambda^2$. Thus the order of $\Lambda$ at an end is invariant.