10 Complete Minimal Surfaces, Osserman's Theorem

Let $X : M \hookrightarrow \mathbf{R}^3$ be a surface, $\Lambda^2 = |X_1|^2 = |X_2|^2$, and $\gamma : I \to M$ be a differentiable curve. The arc length of γ is $\Gamma := \int_I |(X \circ r)'(t)|\, dt$. A *divergent path* on M is a piecewise differentiable curve $\gamma : [0, \infty) \to M$ such that for every compact set $V \subset M$ there is a $T > 0$ such that $\gamma(t) \notin V$ for every $t > T$. If γ is piecewise differentiable, we define its arc length as

$$\Gamma := \int_0^\infty |(X \circ r)'(t)|\, dt = \int_0^\infty \Lambda(\gamma(t))\, |r'(t)|\, dt.$$

Note that Γ could be ∞.

Definition 10.1 We say that X is *complete* if for any divergent curve γ, $\Gamma = \infty$.

Remark 10.2 The use of a divergent curve instead of boundary to describe completeness is because that if $M = \mathbf{D}^*$, $\{0\}$ is not a boundary point of M, but is the limit point of a divergent curve.

Note that in case that (M, g) is a non-compact Riemannian manifold and $\partial M = \emptyset$, according to the Hopf-Reno theorem, this definition of completeness is equivalent to each of the following:

1. Any geodesic $\gamma : I \subset \mathbf{R} \hookrightarrow M$ can be extended to a geodesic $\gamma : \mathbf{R} \hookrightarrow M$,

2. (M, d) is a complete metric space, where d is the induced distance from the Riemannian metric g (roughly speaking, $d(p, q)$ = the arc length of the shortest geodesic segment connecting p and q),

3. in (M, d), any bounded closed set is compact.

In general, there are many examples of closed minimal submanifolds $M \hookrightarrow (N, g)$ where (N, g) is a Riemannian manifold. For example, $S^2 \subset S^3$ is minimal. But we have seen that there are no closed minimal surfaces in $\mathbf{R}^3$. Hence in some sense a complete minimal surface without boundary is the closest analogue to a "closed minimal surface in $\mathbf{R}^3$".

Definition 10.3 Let $X : M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface. Remember that the Gauss curvature K is a non-positive function on M, hence the integral of K has a meaning. We define

$$K(M) := \int_M K dA \qquad (10.39)$$

to be the *total Gauss curvature* of M.

Let $X : M \hookrightarrow \mathbf{R}^3$ be a surface and K be the Gauss curvature. Let $K^- = \max\{-K, 0\}$, $K^+ = \max\{K, 0\}$, then $K = K^+ - K^-$, $|K| = K^+ + K^-$. We first prove a theorem of A. Huber, the proof shown here belongs to B. White [82].