9 Conformal Types of Riemann Surfaces

We will discuss complete minimal surfaces of finite topological type and their annular ends. We need first consider a little of the conformal type of such surfaces.

All *closed* (compact without boundary) 2-dimensional manifolds are classified topologically by their genus and orientability. For example, the topological classification of closed orientable 2-dimensional manifolds is as follows:

The simplest surface is the sphere S^2. Then we can do "surgery" on S^2; by deleting two disjoint disks on S^2 and gluing the boundary of a cylinder along the two circular boundaries we obtain a torus. We say a torus has genus one and is a sphere plus one handle, while S^2 has genus zero, and is a sphere without handle. Thus we obtain genus k surfaces S_k for all integers $k \geq 0$ by adding k handles to a sphere. These are all possible topological types of closed orientable 2-dimensional manifolds. An important topological invariant is Euler's characteristic $\chi(S_k) = 2(1-k)$. Two closed 2-dimensional manifolds are homeomorphic if and only if they have the same Euler's characteristic. Euler's characteristic can be calculated by Gauss-Bonnet Formula, if we have a Riemannian metric on the manifold.

As we have seen before, any smooth orientable 2-dimensional manifold is diffeomorphic to a Riemann surface. Let M and N be two Riemann surfaces. We say that $f : M \to N$ is holomorphic if for any $p \in M$ there is an isothermal coordinate neighbourhood $U \ni p$ with complex coordinate z and an isothermal coordinate $V \ni f(p)$ with complex coordinate w, such that $w \circ f(z)$ is holomorphic.

In the category of Riemann surfaces, $M \cong N$ (*have equivalent conformal type*) if and only if there is a diffeomorphism $f : M \to N$ such that f and f^{-1} are both holomorphic. Such an f is called a *conformal diffeomorphism.*

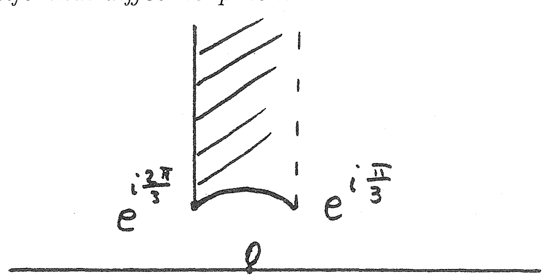

Figure 1

There is considerable interest in classifying the conformal type of closed Riemann surfaces. Although the topological classification is quite simple, the conformal classification is still not clear. In general, S^2 has only one conformal type, i.e., any two *closed* (without boundary) orientable Riemann surfaces of genus zero are conformally diffeomorphic to each other. A typical coordinate system on S^2 is given by stereographic projection from the north and south poles. The conformal structure of genus-one Riemann surfaces corresponds to a region in $\mathbf{C}$ as in the picture above. Such a representation