7 The Geometry of the Enneper-Weierstrass Representation

Let $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface. We will give the geometric data, such as the Gauss map, the first and second fundamental forms, the principal and Gauss curvatures, etc., of a minimal surface via its Enneper-Weierstrass representation.

One important fact is that the meromorphic function g in the Enneper-Weierstrass representation corresponds to the *Gauss map* N. For this we first recall that the Gauss map $N : M \to \Sigma = S^2$ of an immersion $X : M \hookrightarrow \mathbf{R}^3$ is defined as

$$N = |X_u \wedge X_v|^{-1}(X_u \wedge X_v) : M \to \Sigma.$$

Let $\tau : S^2 - \{\mathcal{N}\} \to \mathbf{C}$ be stereographic projection, where $\mathcal{N}$ is the north pole. Then

$$\tau(x,y,z) = \frac{x+iy}{1-z}, \qquad \tau^{-1}(w) = \frac{1}{1+|w|^2}(2\Re w, 2\Im w, |w|^2 - 1),$$

where $\Re$ and $\Im$ are the real and imaginary parts. We claim that

$$g = \tau \circ N : M \to \mathbf{C}.$$

In fact,

$$\tau^{-1} \circ g = \frac{1}{1+|g|^2}(2\Re g, \;\; 2\Im g, \;\; |g|^2 - 1).$$

By (6.15), (6.18), and (6.26)

$$\begin{aligned} X_u &= \Re\left(\frac{1}{2}f(1-g^2), \;\; \frac{i}{2}f(1+g^2), \;\; fg\right), \\ X_v &= -\Im\left(\frac{1}{2}f(1-g^2), \;\; \frac{i}{2}f(1+g^2), \;\; fg\right), \end{aligned}$$

thus

$$X_u \wedge X_v = \begin{pmatrix} -\Re\frac{i}{2}f(1+g^2)\Im fg + \Re fg \Im \frac{i}{2}f(1+g^2) \\ \Re\frac{1}{2}f(1-g^2)\Im fg - \Re fg\Im\frac{1}{2}f(1-g^2) \\ -\Re f(1-g^2)\Im\frac{i}{4}f(1+g^2) + \Re\frac{i}{4}f(1+g^2)\Im f(1-g^2) \end{pmatrix}$$

$$= \begin{pmatrix} \Im[\frac{i}{2}f(1+g^2)\overline{fg}] \\ \Im[\frac{1}{2}\overline{f(1-g^2)}fg] \\ \Im\left[\frac{-i}{4}\overline{f(1+g^2)}f(1-g^2)\right] \end{pmatrix} = \begin{pmatrix} \frac{1}{2}|f|^2\Re(\overline{g} + |g|^2 g) \\ \frac{1}{2}|f|^2\Im(g - |g|^2\overline{g}) \\ \frac{1}{4}|f|^2\Re(|g|^4 - 1 - \overline{g}^2 + g^2) \end{pmatrix}$$