6 The Enneper-Weierstrass Representation

Suppose that $X : M \hookrightarrow \mathbf{R}^3$ is minimal. Since X is harmonic, on an isothermal neighbourhood $(U, (x, y))$,

$$\phi = (\phi_1, \phi_2, \phi_3) = \frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y} = 2\frac{\partial X}{\partial z} \tag{6.15}$$

is holomorphic. In fact,

$$\frac{\partial \phi}{\partial \bar{z}} = 2\frac{\partial^2 X}{\partial \bar{z}\partial z} = \frac{1}{2} \triangle X = \vec{0}.$$

Let V be another isothermal neighborhood with coordinate $w = u + iv$, and let

$$\tilde{\phi} = \frac{\partial X}{\partial u} - i\frac{\partial X}{\partial v}.$$

On $U \cap V$

$$\begin{aligned} \phi = \frac{\partial X}{\partial x} - i\frac{\partial X}{\partial y} &= \frac{\partial X}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial x} - i\left(\frac{\partial X}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial y}\right) \\ &= \left(\frac{\partial X}{\partial u} - i\frac{\partial X}{\partial v}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) = \tilde{\phi}\frac{dw}{dz}. \end{aligned} \tag{6.16}$$

Hence

$$\tilde{\phi}\, dw = \phi\, dz, \tag{6.17}$$

which means that $\phi\, dz$ gives a globally defined vector valued holomorphic 1-form. Write

$$\omega = (\omega_1, \omega_2, \omega_3) = (\phi_1, \phi_2, \phi_3)dz = \phi\, dz. \tag{6.18}$$

By the definition of ϕ, X being conformal is equivalent to

$$\sum_{i=1}^{3} \omega_i^2 = \sum_{i=1}^{3} \phi_i^2 (dz)^2 = 0. \tag{6.19}$$

The condition that X is an immersion is equivalent to

$$\infty > \sum_{i=1}^{3} |\omega_i|^2 = \sum_{i=1}^{3} |\phi_i|^2 |dz|^2 = \left(\left|\frac{\partial X}{\partial x}\right|^2 + \left|\frac{\partial X}{\partial y}\right|^2\right) |dz|^2 = 2\Lambda^2 |dz|^2 > 0. \tag{6.20}$$

Remark 6.1 When $\sum_{i=1}^{3} |\omega_i|^2 = 0$ at some point $p \in M$, we call p a *branch point* of the surface $X : M \to \mathbf{R}^3$. At such a point, X ceases to be an immersion. At times we want to study minimal surfaces with branch points, called *branched minimal surfaces.* For branched minimal surface, since our data ϕ is holomorphic, we see that branch points are isolated. Thus in any precompact domain there are at most a finite number of branch points.

Our main interest is in minimal surfaces without branch points. All minimal surfaces in these notes are branch point free, unless specified otherwise.