5 Isothermal Coordinates for Minimal Surfaces

There is a direct construction of isothermal coordinates for minimal surfaces. Let $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface and $p \in M$. Without of loss generality we can assume that $X(p) = (0,0,0)$ and $N(p) = (0,0,1)$, and there is a simply connected domain $(0,0) \in \Omega \subset \mathbf{R}^2$ such that near $(0,0,0)$, $X(M)$ can be written as a graph $(x, y, u(x,y))$, with $u : \Omega \to \mathbf{R}$ a solution to the minimal surface equation. Writing $p = u_x$, $q = u_y$ and $W = (1+p^2+q^2)^{1/2}$, we see that $pdx + qdy$ is a closed form, i.e., $d(pdx+qdy) = 0$ on Ω. Furthermore, it is also easy to check that the two 1-forms

$$\eta_1 := \frac{1}{W}\left((1+p^2)dx + pq\,dy\right), \qquad \eta_2 := \frac{1}{W}\left(pq\,dx + (1+q^2)dy\right),$$

are closed. Since Ω is simply connected,

$$\xi(x,y) := x + \int_{(0,0)}^{(x,y)} \eta_1 = x + F(x,y), \qquad \eta(x,y) := y + \int_{(0,0)}^{(x,y)} \eta_2 = y + G(x,y),$$

are well defined. Thus

$$\frac{\partial \xi}{\partial x} = 1 + \frac{1+p^2}{W}, \qquad \frac{\partial \xi}{\partial y} = \frac{pq}{W},$$
$$\frac{\partial \eta}{\partial x} = \frac{pq}{W}, \qquad \frac{\partial \eta}{\partial y} = 1 + \frac{1+q^2}{W},$$

and

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = 2 + \frac{2+p^2+q^2}{W} = \frac{(W+1)^2}{W} > 0.$$

Thus the transformation $(x,y) \to (\xi,\eta)$ has a local inverse $(\xi,\eta) \to (x,y)$ and setting $x = x(\xi,\eta)$, $y = y(\xi,\eta)$, $z(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$, we find

$$\frac{\partial x}{\partial \xi} = \frac{W+1+q^2}{(W+1)^2}, \qquad \frac{\partial x}{\partial \eta} = -\frac{pq}{(W+1)^2},$$

$$\frac{\partial y}{\partial \xi} = -\frac{pq}{(W+1)^2}, \qquad \frac{\partial x}{\partial \eta} = \frac{W+1+p^2}{(W+1)^2},$$
$$\frac{\partial z}{\partial \xi} = p\frac{\partial x}{\partial \xi} + q\frac{\partial y}{\partial \xi}, \qquad \frac{\partial z}{\partial \eta} = p\frac{\partial x}{\partial \eta} + q\frac{\partial y}{\partial \eta}.$$

Calculation shows that

$$|X_\xi|^2 = |X_\eta|^2 = \frac{W}{J} = \frac{W^2}{(W+1)^2}, \qquad X_\xi \bullet X_\eta = 0.$$

Thus (ξ,η) is an isothermal coordinate. Furthermore, (ξ,η) has the property that

$$|(\xi,\eta)|^2 > |(x,y)|^2. \tag{5.13}$$