4 The Minimal Surface Equation

Sometimes our surface is a graph over a domain $\Omega \subset \mathbf{R}^2$, i.e., $(x, y, z) \in X(M)$ is expressed as $z = z(x, y)$, $(x, y) \in \Omega$. Moreover, locally we can always treat a "small piece" of surface as a graph. Thus we need know the differential equation governing z, the *minimal surface equation*, in order to derive more information.

To derive the minimal surface equation we use the following equivalent form of $\triangle_X$,

$$\triangle_X X = \sum_{i=1}^{2}[\tau_i\tau_i X - (\nabla_{\tau_i}\tau_i)X], \tag{4.7}$$

where $(\tau_1, \tau_2)(p)$ is an orthonormal frame of T_pM in the induced metric by X and $\nabla_{\tau_i}\tau_i = (D_{\tau_i}\tau_i)^T$ is the covariant differential, in our case, namely the tangent part of $D_{\tau_i}\tau_i$.

Our surface can be written as

$$X(x, y) = (x, y, z(x, y)), \quad (x, y) \in \Omega.$$

Thus $X_x = (1, 0, z_x)$ and $X_y = (0, 1, z_y)$. We will take the upward normal

$$N = \frac{1}{(1 + z_x^2 + z_y^2)^{1/2}}(-z_x, -z_y, 1).$$

We take (τ_1, τ_2) as

$$\tau_1 = dX^{-1}\left(\frac{1}{(1 + z_x^2)^{1/2}}X_x\right) = \frac{1}{(1 + z_x^2)^{1/2}}\frac{\partial}{\partial x},$$

$$\tau_2 = dX^{-1}\left[\left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2}\right)^{1/2}\left(X_y - \frac{z_x z_y}{1 + z_x^2}X_x\right)\right] = \left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2}\right)^{1/2}\left(\frac{\partial}{\partial y} - \frac{z_x z_y}{1 + z_x^2}\frac{\partial}{\partial x}\right).$$

By (4.7) and (2.1),

$$\begin{aligned} 2H &= (D_{\tau_1}\frac{1}{(1 + z_x^2)^{1/2}}\frac{\partial}{\partial x} + D_{\tau_2}\left[\left(\frac{1 + z_x^2}{1 + z_x^2 + z_y^2}\right)^{1/2}\left(\frac{\partial}{\partial y} - \frac{z_x z_y}{1 + z_x^2}\frac{\partial}{\partial x}\right)\right] \bullet N \\ &= \left[\frac{X_{xx}}{1 + z_x^2} + \frac{z_x^2 z_y^2 X_{xx}}{(1 + z_x^2)(1 + z_x^2 + z_y^2)} - \frac{2z_x z_y X_{xy}}{1 + z_x^2 + z_y^2} + \frac{(1 + z_x^2)X_{yy}}{1 + z_x^2 + z_y^2}\right] \bullet N \\ &= \left[\frac{1 + z_y^2}{1 + z_x^2 + z_y^2}X_{xx} - \frac{2z_x z_y}{1 + z_x^2 + z_y^2}X_{xy} + \frac{1 + z_x^2}{1 + z_x^2 + z_y^2}X_{yy}\right] \bullet N. \end{aligned}$$

Since $X_{xx} = (0, 0, z_{xx})$, $X_{xy} = (0, 0, z_{xy})$, and $X_{yy} = (0, 0, z_{yy})$, we have

$$2H = \frac{1}{(1 + z_x^2 + z_y^2)^{3/2}}\left[(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy}\right].$$