2 Definition of Minimal Surfaces

Definition 2.1 A *minimal surface* in $\mathbf{R}^3$ is a *conformal harmonic immersion* $X : M \hookrightarrow \mathbf{R}^3$, where M is a 2-dimensional smooth manifold, with or without boundary. Here *conformal* means that for any point $p \in M$ there is a local coordinate neighbourhood $(U, (u, v))$ on M, such that in U the vectors

$$X_1 := X_u = \frac{\partial X}{\partial u} = \left(\frac{\partial X^1}{\partial u}, \frac{\partial X^2}{\partial u}, \frac{\partial X^3}{\partial u}\right) = (X_u^1, X_u^2, X_u^3) = (X_1^1, X_1^2, X_1^3)$$

and

$$X_2 := X_v = \frac{\partial X}{\partial v} = \left(\frac{\partial X^1}{\partial v}, \frac{\partial X^2}{\partial v}, \frac{\partial X^3}{\partial v}\right) = (X_v^1, X_v^2, X_v^3) = (X_2^1, X_2^2, X_2^3)$$

are perpendicular to each other and have the same length. Thus

$$\Lambda^2 := |X_u|^2 = |X_v|^2 > 0, \quad X_u \bullet X_v \equiv 0.$$

Here $\bullet$ is the Euclidean inner product. Such a coordinate neighbourhood $(U, (u, v))$ is called an *isothermal neighbourhood,* its coordinates (u, v) are called *isothermal coordinates.*

The word *immersion* means that for any $p \in M$, $X_* := dX : T_pM \to T_{X(p)}\mathbf{R}^3$ is a linear embedding. In the case X is conformal, it means simply that $\Lambda > 0$ on M.

The word *harmonic* means that

$$\triangle X = \frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2} = X_{uu} + X_{vv} = X_{11} + X_{22} \equiv \vec{0}.$$

If M is connected, then we say that the surface X is *connected.* We will only consider connected surfaces. Furthermore, since any non-orientable surface has an orientable double covering, we will only consider *oriented minimal surfaces.*

A homothety of $\mathbf{R}^3$ is the composition of a rigid motion and a dilation or a shrinking. Let T be a homothety of $\mathbf{R}^3$, $X : M \hookrightarrow \mathbf{R}^3$ be a surface. It is easy to see that X is a conformal harmonic immersion if and only $T \circ X$ is. Thus we consider all surfaces in $\mathbf{R}^3$ up to a homothety. That is, we do not distinguish the surfaces $X : M \hookrightarrow \mathbf{R}^3$ and $T \circ X : M \hookrightarrow \mathbf{R}^3$.

A classical theorem says that any C^k immersion, $2 \leq k \leq \infty$, can have an atlas of isothermal coordinate charts, so that X being conformal is not a special property of minimal surfaces. The important fact which distinguishes minimal surfaces is that under these isothermal charts, X is harmonic.

For an orientable surface $X : M \hookrightarrow \mathbf{R}^3$, let $\{(U_\alpha, z_\alpha = u_\alpha + iv_\alpha)\}_{\alpha \in A}$ be an atlas of isothermal coordinates of the same orientation, then $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ defines a *complex (conformal) structure* on M.