SOLITON GEOMETRY, KAC-MOODY ALGEBRAS AND THE YANG-BAXTER EQUATIONS

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1.INTRODUCTION AND EXAMPLES

This note is based on joint work with D.Ferus, F. Pedit and U.Pinkall [1].

Soliton geometry has been successfully applied to the study of linear Weingarten surfaces (e.g. $H=\frac{1}{2}$, $K=-1$, smoke rings (tubular surfaces), minimal, flat) in three dimensional Riemannian and Lorentzian space forms, minimal surfaces in S^4, minimal surfaces in the Lorentzian S^4 (which are almost in 1:1 correspondence with Willmore surfaces), harmonic maps in Lie groups (and symmetric spaces), isometric imbeddings $M^n(k) \to M^{2n-1}(k')$, etc.

Since we will be discussing a quite general method which applies to all the above examples it is important to also consider a specific case in order to get a feel for the difficulties involved in "setting the dials on the machine". The case we'll consider is that of minimal surfaces in S^4. For the sake of completeness we'll first review the classical cases to show how soliton PDE's arise natually in geometry.

2. NICE SURFACES COME FROM SOLITON EQUATIONS

Let H and K be the mean and Gauss curvatures.

If $f:\mathbb{R}^2 \to \mathbb{R}^3$ has $K=-1$ and is parametrized by asymptotic lines with $|f_x| = |f_y| = 1$ and $\omega(x,y)$ is the angle between the asymptotic lines (i.e. arccos $<fx,fy>$), then ω satisfies the sine-Gordon equation $\omega_{xy}= \sin \omega$.

If $f:\mathbb{R}^2 \to \mathbb{R}^3$ has $H=\frac{1}{2}$ and is parametrized by curvature lines with $<f_z,f_z> = 0$ (i.e. f is conformal) and $\omega(x,y)$ is $\log|f_x|$, the ω satisfies the sinh-Gordon equation $\omega_{z\bar{z}} + \frac{1}{2}\sinh(2\omega) = 0$.