Second Order Elliptic Equations with Venttsel Boundary Conditions

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1 Introduction

In this work we study the following boundary value problems. Let $\Omega \subset R^n$ be a bounded domain with smooth boundary. We define the tangential gradient operator on the boundary manifold $\partial\Omega$ by $\delta_i = D_i - \nu_i\nu_k D_k$, where $\nu = (\nu_1, \ldots, \nu_n)$ is the inner normal vector to $\partial\Omega$. We consider the problem

$$Lu \equiv a^{ij}D_{ij}u + b^iD_iu + cu = f, \qquad \text{in } \Omega, \tag{1}$$

$$Bu \equiv \alpha^{ij}\delta_i\delta_j + \beta^iD_iu + \gamma u = g, \qquad \text{on } \partial\Omega. \tag{2}$$

The equation (1) is elliptic in the usual sense [2] and the boundary condition (2) is called Venttsel if the following conditions are satisfied:

$$\alpha^{ij}\eta_i\eta_j \geq 0, \qquad \text{for } \eta \in R^n \text{ s.t. } \eta \perp \nu, \tag{3}$$

and

$$\beta^i\nu_i \geq 0. \tag{4}$$

If the inequality in (3) is strict the boundary condition B is called elliptic, otherwise it is called degenerate elliptic; if the inequality in (4) is strict the boundary condition B is called oblique, otherwise it is called degenerate oblique.

Without loss of generality we may assume that $\{\alpha^{ij}\}$ has the property

$$\alpha^{ij}\nu_j \equiv 0, \qquad \text{on } \partial\Omega, \text{ for } i = 1, 2, \ldots, n.$$

Therefore when we locally flatten the boundary our boundary value problem becomes

$$a^{ij}D_{ij}u + b^iD_iu + cu = f, \qquad \text{in } B^+, \tag{5}$$

$$\alpha^{st}D_{st}u + \beta^iD_iu + \gamma u = g, \qquad \text{on } B^0 \tag{6}$$

where $B^+ = \{x \in R^n |\ |x| < 1, x_n > 0\}$,$B^0 = \{x \in R^n |\ |x| < 1, x_n = 0\}$, and the repeated indices s, t indicate the summation from 1 to $n-1$.

This type of boundary condition originally came from probability theory (see for example Venttsel [10]), and also occurs in three-dimensional water wave theory, (see Shinbrot [9]),