ELLIPTIC SYSTEMS

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1 Introduction

Elliptic equations model the behaviour of *scalar* quantities u, such as temperature or gravitational potential, which are in an equilibrium situation subject to certain imposed boundary conditions. In his first four lectures, John Urbas discussed *linear*[1] elliptic equations. In his lectures on the minimal surface equation, Graham Williams discussed the minimal surface equation, a *quasilinear*[2] elliptic equation in divergence form. Neil Trudinger and Tim Cranny will discuss *fully nonlinear*[3] elliptic equations.

Elliptic systems model *vector-valued* quantities in an equilibrium situation subject to certain imposed boundary conditions. Examples are a vectorfield describing the molecular orientation of a liquid crystal, and the displacement of an elastic body under an external force.

Solutions of elliptic *equations* are typically as smooth as the data allows (e.g. are C^∞ if the given data is C^∞). Solutions of elliptic *systems* typically have singularities.

We use as reference [G] the book *Multiple Integrals in the Calculus of Variations* by M. Giaquinta.

2 A Model, Harmonic Map, Problem

Suppose $\Omega \subset \mathbb{R}^n$ is an elastic membrane, "stretched" via the function w over a part of the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$, where w is specified on the boundary $\partial\Omega$. As a simple approximation to the physical situation, we can regard w as a minimiser of the *Dirichlet energy*

$$\frac{1}{2}\int_\Omega |Dw|^2,^4 \qquad (1)$$

amongst all maps $w:\Omega \to \mathbb{R}^{n+1}$ such that

$$|w| = 1, \quad w|_{\partial\Omega} \text{ specified.}$$

[1]The unknown function u and its first and second derivatives occur linearly. The *coefficients* of u and its derivatives may be nonlinear, but usually smooth, functions of the domain variables $x_1, \ldots, x_n$.

[2]Linear in the second derivatives of u, but not necessarily linear in u or its first derivatives.

[3]Not even linear in the second derivatives of u.

[4]Where $|Dw|^2 = \sum_{i,\alpha} |D_i w^\alpha|^2$. The $\frac{1}{2}$ is merely a convenient normalisation constant.