FORMATION OF SINGULARITIES IN SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION WITH CRITICAL POWER NONLINEARITY

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1. INTRODUCTION AND RESULTS

In this paper, the author would like to report his results of recent papers [23–28] concerning the nonlinear Schrödinger equation with critical power nonlinearity (NSC). Our main results are Theorem A ("L^2 concentration" phenomena [23, 24 , 26]), Theorem B (Asymptotics of blow-up solutions [27, 28]) and Theorem C (existence of "blow-up" solutions in the energy space $H^1(\mathbb{R}^N)$ [27, 28]). Moreover, in Sect. 4, he shall briefly mention the further results.

We start with a review of the Cauchy problem for the nonlinear Schrödinger equation:

$$\text{C(p)} \qquad \begin{cases} \text{(NS)} \quad 2i\dfrac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ \text{(IV)} \quad u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Here $i = \sqrt{-1}$, $u_0 \in H^1(\mathbb{R}^N)$ and Δ is the Laplace operator on $\mathbb{R}^N$. The unique local existence of solutions of C(p) is well known for $1 < p < 2^* - 1$ ($2^* = \frac{2N}{N-2}$ if $N \geqq 3$, $= \infty$ if $N = 1, 2$): For any $u_0 \in H^1(\mathbb{R}^N)$, there exist a unique solution $u(t,x)$ of C(p) in $C([0,T_m); H^1(\mathbb{R}^N))$ for some $T_m \in (0,\infty]$ (maximal existence time), and $u(t)$ satisfies the following three conservation laws of L^2, the energy and the momentum:

$$\|u(t)\| = \|u_0\|, \tag{1.1}$$

$$E_{p+1}(u(t)) \equiv \|\nabla u(t)\|^2 - \frac{2}{p+1}\|u(t)\|_{p+1}^{p+1} = E_{p+1}(u_0), \tag{1.2}$$

$$\Im \int_{\mathbb{R}^N} \nabla u(t,x)\overline{u(t,x)}dx = \Im \int_{\mathbb{R}^N} \nabla u_0(x)\overline{u_0(x)}dx \tag{1.3}$$

for $t \in [0,T_m)$, where $\|\cdot\|$ and $\|\cdot\|_{p+1}$ denotes the L^2 norm and L^{p+1} norm respectively. Furthermore $T_m = \infty$ or $T_m < \infty$ and $\lim_{t\to T_m} \|\nabla u(t)\| = \infty$. For details, see, *e.g.*, [11,12,14].

As for the existence and non-existence of global solutions of C(p), the following is well known.

(i) If $1 < p < 1 + \frac{4}{N}$, there exists a global solution $u \in C_b(\mathbb{R}; H^1(\mathbb{R}^N))$, for any $u_0 \in H^1(\mathbb{R}^N)$, where $C_b(\mathbb{R}; H^1(\mathbb{R}^N)) = C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}; H^1(\mathbb{R}^N))$. See [11,12,14].

(ii) If $1 + \frac{4}{N} \leqq p < 2^* - 1$, there is a subset $\mathcal{B} \in H^1(\mathbb{R}^N)$ such that for any $u_0 \in \mathcal{B}$ the solution of C(p) blows up, *i.e.* the L^2 norm of its gradient explodes in finite time T_m. See [13,29,30,31,36].