On Mean Curvature Flow of Surfaces in Riemannian 3-Manifolds

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In this talk I would like to consider hypersurfaces $(M_t)_{t\in[0,T)}$ which move by mean curvature in a Riemannian manifold N^{n+1}. The hypersurfaces are described by a one-parameter family

$$x_t = x(\cdot, t) : M^n \to N^{n+1}$$

of smooth immersions of an n - dimensional manifold M^n without boundary, with images $M_t = x_t(M^n)$, satisfying

$$\frac{d}{dt}x(p,t) = -H(p,t)\,\nu(p,t) \qquad p \in M^n\,,\ t \in (0,T). \tag{1}$$

Here $H(p,t)$ and $\nu(p,t)$ denote mean curvature and a choice of unit normal of the hypersurface M_t at $x(p,t)$.

Mean curvature flow arises naturally as the steepest descent flow for the area functional which explains its great significance due to its possible applications to minimal surface theory in Riemannian manifolds. Indeed, from (1) one easily derives the evolution equation for the surface element of M_t,

$$\frac{d}{dt}\mu_t = -H^2\mu_t \tag{2}$$

see [H1], which immediately yields

$$\mathcal{H}^n(M_t) \leq \mathcal{H}^n(M_0) \tag{3}$$

for all $t \geq 0$ where $\mathcal{H}^n$ denotes n - dimensional Hausdorff-measure.

Without any special assumptions on M_0, such as convexity (see [H1], [H2]), a solution of (1) will in general develop singularities in finite time before it 'disappears', as for example Grayson's axially symmetric dumbbell surface ([Gr]).

In [B], K. Brakke has studied the regularity behaviour of n - dimensional varifolds moving by mean curvature in $\mathbf{R}^{n+k}, k \geq 1$. For smooth solutions of (1) with $T < \infty$, his result implies that if there exists a unit density rectifiable varifold M_T such that $M_t \to M_T$ as $t \to T$ in the measure-theoretic sense in $B_\rho(x_0)$ for some $x_0 \in \mathbf{R}^{n+1}$