Appendix F

Bifurcating geodesics in $C^{1,\alpha}_{loc}$, $0 < \alpha < 1$, metrics.

In this Appendix we present $C^{1,\alpha}_{loc}$, $0 < \alpha < 1$, metrics for which there exist bifurcating spacelike, timelike or null geodesics. From the classical theory of ODE's it is well known that the initial value problem (IVP) for the geodesic equation is uniquely solvable when the metric is $C^{1,1}_{loc}$. Due to the variational character of the geodesic equations one could hope, at least in the strictly Riemannian case, to have uniqueness of the IVP for geodesics under some weaker conditions: the examples of this Appendix show that the requirement of $C^{1,1}_{loc}$ differentiability of the metric cannot be relaxed without introducing some supplementary conditions. The examples presented here have been worked out in collaboration with J.Isenberg, following a suggestion by R.Hamilton[1].

Let us start by noting that if we have a metric which is $C^{1,1}_{loc}$ except possibly at an isolated point x_o in a neighborhood of which

$$|g(x)_{ij} - \delta_{ij}| \leq Cr(x - x_o)^\alpha, \quad |\partial_k g_{ij}(x)| \leq Cr(x - x_o)^{\alpha-1},$$

$$|\partial_k \partial_l g_{ij}(x)| \leq Cr(x - x_o)^{\alpha-2}, \quad 0 < \alpha < 1, \tag{F.0.1}$$

uniqueness of the initial value problem for geodesics can be established by standard fixed point methods in an appropriately weighted space of functions (recall that existence

[1]It has been pointed out to the author by R. Bartnik, that essentially the same example has been presented in [65] for spacelike geodesics in a Riemannian metric.