

Appendix E

On a wave equation with a singular source.

In this Appendix we shall show that the solutions of the problem (1.8.3)–(1.8.4), with $0 \notin \text{supp } f$, $f \in C^\infty(\mathbb{R})$, $\varphi, \psi \in C^\infty(\mathbb{R}^2)$, are smooth on $\mathbb{R}^3 \setminus \text{supp } \rho$. Consider thus the equation

$$\square U = f(t) \delta_0,$$

recall that

$$\Delta \ln r = 2\pi \delta_0, \tag{E.0.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and we also have

$$0 < \alpha \in \mathbb{R}, \quad \Delta r^\alpha \ln r = \alpha(\alpha \ln r + 2) r^{\alpha-2}. \tag{E.0.2}$$

From (E.0.1) one finds that the function $U_1 = U - f(t) \ln r / 2\pi$ satisfies

$$\square U_1 = -\frac{1}{2\pi} \frac{\partial^2 f}{\partial t^2} \ln r,$$

and using (E.0.2) one shows by induction that there exist functions $\varphi_i \in C^\infty(\mathbb{R})$, $0 \notin \text{supp } \varphi_i$, such that for any $k \in \mathbb{N}$ we have

$$\begin{aligned} \square U_k &\equiv \square \left(U - \sum_{i=0}^k \varphi_i(t) r^{2i} \ln r \right) \\ &= \chi_k(t) r^{2k} \ln r + \psi_k \equiv \rho_k, \end{aligned}$$